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A BILLET ON MULTI-PANTOGRAPH ORDINARY DIFFERENTIAL EQUATIONS WITH CONSTANT DEVIATING ARGUMENTS: ISSUES ON THE ROLE OF LÉVY NOISE IN THE ALMOST SURE EXPONENTIAL STOCHASTIC-SELF STABILIZATION

[1*]Nwoye, F.C., [2]Atonuje, E.O. and [3]Atonuje, A.O.

[1,3]Department of Mathematics Delta State University, Abraka, Nigeria.,

[2]Department of Computer Science, Delta State University, Abraka, Nigeria.

*Corresponding Author: faithnwoye@delsu.edu.ng

ABSTRACT

This letter considers the role played by Lévy noise in the almost sure exponential stochastic self-stabilization of solution to a multi-pantograph delay differential equation (MPDDE). The equation is nonlinear and contains multi-pantograph terms as well as several constant time lags and as such, the solution is generally unstable. The system is perturbed by random noise disturbance of Lévy -type. It is established that under certain conditions, the presence of Lévy noise would stochastically self-stabilize the resulting stochastic multi-pantograph differential equation in an almost sure exponential sense, whereas the comparable deterministic multi-pantograph delay differential equation, in which Lévy noise is absent, still remains unstable.

Key words: Levy noise, Exponential stabilization, Pantograph, Differential Equation

INTRODUCTION

Differential equations are used to desribe a relationship between functions and their derivatives. When physical processes do not depend on only the current state, they are easily represented by means of delay differential equations (DDEs). The significance of these equations lies in their ability to describe processes with aftereffects (see Rhihan (2018), Unser (2020), Jordan and Turkington (2001), Atonuje and Ayoola (2007). Situations in life are not usually free of disturbances which may be intrinsic or external. Most real-world dynamic systems are modeled by deterministic differential equations when not influenced by randomness, but by changes in state or parameters due to deterministic forces. Whenever such changes exist, the systems are said to be chaotic. Where real life situations are subject to internal or external disturbances, they are better modeled mathematically by means of stochastic delay differential equations (SDDEs). How to establish the combined roles of time lags and Ito-type noise disturbances in creating, sustaining and disrupting oscillatory behavior of solutions to deterministic delay differential equations is the subject of the articles of Atonuje and Ayoola (2008), Atonuje (2010) and Atonuje and Ayoola (2007). Papers on the stability in solutions of stochastic differential equations were written by Njoseh and Atonuje (2006), Atonuje and Ezenweani (2011), Atonuje and Apanapudor (2017), Zhang *et al.*, (2019), Wu, *et al.*, (2010) and Zhu *et al.*, (2017).

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Among other works, there exists a collection of literature that indicates application areas of ordinary differential equations (ODEs), delay differential equations (DDEs) and stochastic delay differential equations (SDDEs) and we cite Gopalsamy (1987), Jordan (2008), Atonuje (2010), Ezzinbi and Kad (2006), Goriety and Hiyde (1998), Mao (1997), Applebaum and Siakali (2009), Applebaum (2004), Agusto *et al.*, (2017), Atonuje and Tsetimi (2016), Arnold (1974), Baker and Buckwar (2000) and Kloeden and Platen (1992).

Pantograph differential equations are variant delay differential equations which have been widely studied since the 1850s. According to Rubab and Ahmad (2016), the pantograph is a device installed on the electric locomotive. The first electric locomotive was made in 1851 and became commissioned in 1895. Taylor and Ockendon (1971) studied how electric current is collected by the pantograph of an electric locomotive and then developed a mathematical model of pantograph. Functional differential equations with proportional delays are thus usually referred to as pantograph equations.

The importance of pantograph differential equations of the type studied here emanates from the key role they play in understanding the asymptotic behavior and stability properties of complex systems such as electric locomotives, electrodynamics, astrophysics, nonlinear dynamic systems, electronic systems, population dynamics, probability theory on algebraic structures and quantum mechanics (Florescu (2014), Fristedt and Gray (2013) and Sezer *et al.*, (2008). They are also applicable in computer science for image processing, signal processing, Simulations, sensor data, Neural networks, (see Unser (2020), Jordan and Turkington (2001)).

A key focus in the study of differential equations is the concept of stability which was first revealed in the research effort of A. M. Lyapunov in 1892. Since then, the mathematical community and numerical analysts have been pre-occupied with the study of stability. See for instance the works Alferov *et al.*, (2018), Gao *et al.*, (2013). Motivated by some of the works of the authors mentioned above, we focus our attention on the study of stability behavior of deterministic multi-pantograph ordinary differential equation:

$$x'(t) = f\left(t, x(t), x(t-r_i)x(q(t),)x\big(q_2(t)\big), x\big(q_3(t)\big), \dots, x\big(q_m(t)\big)\right), \quad t \ge 0, x(0) = \lambda \tag{1}$$

Where $0 < q_i < 1, i = 1, 2, \dots, m$ are the pantograph functions, $r_i, i = 1, 2, \dots, m$ are constant deviating arguments and f is a continuous function so that Eq. (1) is generally unstable (Mao (1997)). Eq. (1) is a functional differential equation that was first used for current collection in an electric locomotive system. [Ahmad and Mukhtar (2015)].

A lot of research articles and monographs have been written on stability of ODEs and DDEs. Many of the research articles focused on providing necessary and sufficient conditions for solutions of such equations to be stable, either in probability sense, mean square, asymptotically, p^{th} mean sense, in bifurcating periodic solutions or in an almost sure exponential stability as well as stability in numerical schemes. A search through these articles shows that the study of the effects of Lévy noise in stochastically stabilizing dynamic systems is scantily researched or is entirely missing from existing literature.

In the present article, we focus our attention on providing an answer to the following questions: if under certain conditions, a deterministic pantograph delay differential equation which is generally unstable, is stochastically perturbed by Lévy noise, can the presence of a sufficiently strong noise driving force parameter stochastically stabilize the system in an almost sure (a.s.) exponential sense? Under these same conditions, can the comparable deterministic system where noise is absent be stabilized or would it remain unstable?

This research is thus focused on proposing to establish the contribution of Lévy noise to the almost sure exponential stochastic self-stabilization in a specific type of deterministic pantograph delay differential equation. This approach was first suggested by Mao (1997) and it utilizes the concept of Lyapunov sample exponent function.

METHODOLOGY AND PRELIMINARIES

One of the most useful deterministic models which appear frequently in application to electric locomotive and other dynamic systems is the multi-pantograph delay differential equation.

The present article studies the deterministic ordinary delay multi-pantograph differential equation of the form:

$$\left.\begin{aligned} x'(t) &= \lambda x(t) + \sum_{i=1}^{k} \mu_i f\big(t, x(t), x(t-r_i), x(q_i t)\big), t \in [0,T] \\ x(t) &= \varphi(t), t \in [-\Gamma, 0] \end{aligned}\right\} \qquad (2)$$

Where, $f(.)$ is an analytic function, for $x > 0$, $r_i > 0$ are fixed delays, $\lambda, \mu_i \in \mathbb{R}^+$, $q_i < q_{i-1} < q_{i-2} < 1$, $t > 0$ *are the pantograph terms* , $\varphi \in C([-\Gamma, 0], \mathbb{R}^d)$ is the initial function and $\Gamma = \max_{0\le i\le 1}\{r_i\}$. The equation is nonlinear and contains a multi-pantograph term as well as several constant time lags and as such, the solution is generally unstable Mao (1997), Hahn (1967).

By solution of Eq.(2), we refer to a continuous function $x \in C([\bar{t} - \Gamma, \infty), \mathbb{R})$ for some $\bar{t}$ which satisfies Eq. (2) together with its initial function for $t \geq \bar{t}$. Assume that for every initial datum $x(t_0) = \varphi(t_0) \in \mathbb{R}^d$, there exists a global solution $x(t, t_0, x_0)$. Assume also that $f\big(t, x(t_0 - r_i), x(q_i t)\big) = 0, \forall\, t \geq t_0$ so that the solution $x(t, t_0, x_0) \equiv 0$ corresponding to the initial datum $x(t_0) = \varphi(t_0)$. This solution is called the trivial solution or equilibrium position of Eq. (2).

The trivial solution $x(t, t_0, x_0)$ of Eq.(2) is stable if for every $\varepsilon > 0$, there exists $\delta = \delta(t_0, \varepsilon) > 0$ *such that* $\left|x\big(t, t_0, x_0\big)\right| < \varepsilon, \forall\, t \geq t_0$ *and* $|x_0| < \delta$.

Eq. (2) is now perturbed by Levy noise into a stochastic multi-pantograph differential equation with several deviating arguments of the form:

$$\begin{cases} dx(t) = \left(\lambda x(t) + \sum_{i=1}^{n} \mu_i f\big(t, x(t), x(t-r_i), x(q_i t)\big)\right) dt + \\ \sigma\left(\left[\sum_{k=1}^{m} G_k x(t) dB_k(t)\right] + \int_{|y|<\Delta} D(y)\, x(t) \bar{N}(dt, dy) + \int_{|y|\geq\Delta} E(y)\, x(t) N(dt, dy)\right) \\ x(t) = \varphi(t), t \in [-\Gamma, 0] \end{cases} \qquad (3)$$

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For all $t \geq t_0$, where $G_k \in M_d(\mathbb{R})\ for\ 1 \leq k \leq m$, D and E are suitable functions. $D: \mathbb{R}^m \to M_d(\mathbb{R})$, $E: \mathbb{R}^m \to M_d(\mathbb{R})$. σ is the noise driving force parameter which measures the intensity of the fast fluctuating Levy noise. The positive number Δ plays the role of separating small jumps (which are compensated) from large jumps (which are not compensated). N(.) is the Poisson process with compensation $\bar{N}$ and $B_k(t) = (B_1(t),\ B_2(t), \dots, B_m(t)\ )^T, t \geq 0$ is an m-dimensional Brownian motion defined on a complete probability triple (Ω, F, P), with filtration $\{F_t\}_{t\geq 0}$ which is right continuous and each F_t contains all p-null sets in F with expectation $E(x) = \int_\Omega x dP$, where ε denotes expectation.

Recall that every Levy process is the sum of a Brownian motion with drift and another independent random variable. In the above case the other random process is Poisson. This stochastic sum is called the Levy-Ito decomposition and is referred to as a stochastic sum of independent Poisson random variable.

The following assumptions are imposed on the functions $f, D, E\ and\ \varphi$:

i) $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}, D: \mathbb{R} \times \mathbb{R} \to \mathbb{R}, E: \mathbb{R} \times \mathbb{R} \to \mathbb{R}\ and\ \varphi: [-\Gamma, 0] \to \mathbb{R}$ are such that f, D and E are continuous functions and fulfill the uniform Lipchitz condition. More specifically, there exist some positive constants K_1, K_2, K_3, K_4 K_5 and K_6 such that $\forall\ a_1, a_2, b_1, b_2, \in \mathbb{R}$

$$|f(a_1, b_1) - f(a_2, b_2)| \leq K_1\,|a_1 - a_2| + K_2\,|b_1 - b_2| \tag{4}$$
$$|D(a_1, b_1) - D(a_2, b_2)| \leq K_3\,|a_1 - a_2| + K_4\,|b_1 - b_2| \tag{5}$$
$$|E(a_1, b_1) - E(a_2, b_2)| \leq K_5\,|a_1 - a_2| + K_6\,|b_1 - b_2| \tag{6}$$

ii) $f, D\ and\ E$ fulfill the linear growth bound condition. More specifically, there exist positive constants $L_1, L_2\ and\ L_3$ such that for all $a, a_1, b, b_1, c, c_1 \in \mathbb{R}$ such that

$$|f(a, a_1)|^2 \leq L_1(1 + |\,a\,|^2 + |\,a_1|^2) \tag{7}$$
$$|D(b, b_1)|^2 \leq L_2(1 + |\,b\,|^2 + |\,b_1|^2) \tag{8}$$
$$|E(c, c_1)|^2 \leq L_3(1 + |\,c\,|^2 + |\,c_1|^2) \tag{9}$$

iii) The initial function φ is Holder-continuous with exponent β. More specifically, there exists a positive constant K_6 such that $\forall\ t, s \in [-\Gamma, 0]$, $\varepsilon(|\varphi(t) - \varphi(s)|^p) \leq K_6\,|t - s|^{p\beta}, p = 1, 2.$ (10)

The conditions (i) and (ii) ensures the existence and uniqueness of solutions of Eq. (3). For detailed understanding of the concept of expectation, existence and uniqueness of solutions of differential equations, the reader is referred to Arnold (1974).

By solution of the SPDDE (3), one refers to an $\mathbb{R} - value\ x(t): [-\Gamma, 0] \times \Omega \to \mathbb{R}$, called a strong solution of Eq. (3) if it is a measurable sample – continuous process such that $x|[0, T]$ is $\{F_t\}_{0\leq t\leq T} - adapted$, $f, D\ and\ E$ are continuous functions and $x(t)$ satisfies Eq. (3) almost surely as well as the initial condition $x(t) = \varphi(t)$, $t\epsilon\left[-\Gamma, 0\right]$. $x(t)$ is aid to be unique if there exists any other solution $\bar{x}(t)$ is indistinguishable from $x(t)$. that is, $P(x(t) = \bar{x}(t), for\ all\ t \in [-\Gamma, 0]) = 1$.

Eq. (3) can be stabilized in an almost sure (a.s.) exponential sense by Levy noise provided that the noise driving force parameter σ is sufficiently large, i.e., $\sigma < \infty$.

In this section, a carefully chosen integral expression is used to replace σ. that is, $\sigma = \int_0^t |\eta(s)x(s)|^p ds, p \in \mathbb{R}_+$, where $\eta(s)$ is a continuous $\mathbb{R}^{m\times d} - valued$ function with the property $\|\eta(s)\| \le Qe^{\delta t}, \forall t \ge 0$. $\eta(s)$ acts to restrain the solution $x(t)$ so that it must not vanish into infinity at t. $\eta(t)$ in this capacity is called a convergence rate function.

$$\begin{cases} dX(t) = \left(\lambda X(t) + \sum_{i=1}^{n} \mu_i f\left(t, X(t), X(t-r_i), X(q_i t)\right)\right) dt + \\ \left(\int_0^t |\eta(s)x(s)|^p ds\right)\left(\left[\sum_{k=1}^{m} G_k X(t) dB_k(t)\right] + \int_{|y|<\Delta} D(y)\, x(t)\bar{N}(dt, dy) + \int_{|y|\ge\Delta} E(y)\, x(t) N(dt, dy)\right) \\ X(t) = \varphi(t), t \in [-\Gamma, 0] \end{cases} \tag{11}$$

That is, the noise driving force parameter σ in Eq. (3) is carefully replaced by $\int_0^t |\eta(s)x(t,w)|^p ds$, where $\eta(s)$ is the convergence rate function. For Eq. (11) to be almost surely exponentially self-stabilized, we impose the condition that for some $w \in \Omega, \sigma = \int_0^t |\eta(s)x(t,w)|^p ds < \infty$. This means that the noise driving force parameter must be finite.

Definition 1:

The trivial solution $x(t;\ t_{0,}x_0)$ of Eq. (3) is said to almost surely (a. s.) exponentially stable if $\underset{t\to\infty}{LimitSup} \frac{1}{t} log|x(t;\ t_0, x_0)| < 0, a.s. for\ all\ x_0 \in \mathbb{R}^d$.

where, $\underset{t\to\infty}{LimitSup} \frac{1}{t} log|x(t;\ t_0, x_0)|$ is called 'the Lyapunov sample exponent' function.

The main focus of this work is to carefully choose a noise scaling parameter $\sigma < \infty$ and derive results which ensure an a.s. exponential stability of Eq. (3). This is only possible due to the presence of Levy noise. The comparable deterministic Eq. (2), where Levy noise is absent still remains unstable in general. Almost sure exponential Stability is here induced by the presence of Levy noise.

THE MAIN RESULTS

In this section, we derive results which show that the presence of Levy noise in the stochastically perturbed multi-pantograph Eq. (3) ensures that the equation is almost surely exponentially stable under certain imposed conjectures and lemmas.

Denote by L, the differential operator connected with Eq. (3). Let $0 < c \le \infty$ so that $C^{2,1}(\mathbb{R}_+ \times [t_0, \infty), \mathbb{R}_+)$ be the class of all positive functions $\mathrm{H}(x,t)$ defined on $\mathbb{R}_+ \times [t_0, \infty)$ which are continuously twice

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differentiable in x and once differentiable in t. Then L is defined by

$$L = \frac{\delta}{\delta t} + \sum_{i=1}^{d} f_i(x,t)\frac{\delta}{\delta x_i} + \frac{1}{2}\sum_{i,j=1}^{d} |G_k^T(x,t)G_k(x,t)||D^T(x,t)D(x,t)||E^T(x,t)E(x,t)|_{i,j}\frac{\delta t}{\delta x_i \delta x_j}$$

Assume that L acts on $\mathrm{H}(x,t) \in C^{2,1}(\mathbb{R}_+ \times [t_0,\infty), \mathbb{R}_+)$, then

$$LH(x(t),t) = H_t(x.t) + H_t(x.t)f(x,t)$$
$$+\frac{1}{2}trace\left[\left(G_k^T(x,t)H_{xx}(x,t)\right)\left(D^T(x,t)H_{xx}(x,t)(x,t)\right)\left(E^T(x,t)H_{xx}(x,t)(x,t)\right)\right]$$

The following results are useful in this section:

Theorem 3.1:

Assume that there exists a function $\mathrm{H} \in C^{2,1}(\mathbb{R}^d \times [t_0,\infty), \mathbb{R}_+)$ and constants $p > 0, h_1 > 0, h_2 \in \mathbb{R}, h_3 \geq 0$ such that for all $x \neq 0$ and $t \geq t_0$, the following conditions hold:

$$(i)\ h_1|x|^p \leq \mathrm{H}(x,t)$$
$$(ii)\ L\mathrm{H}(x,t) \leq h_2\mathrm{H}(x,t)$$
$$(iii)\ |\mathrm{H}_x(x,t)g(x,t)|^2 \geq h_3\,\mathrm{H}^2(x,t)$$

Then, $\underset{t\to\infty}{LimitSup}\frac{1}{t}log|X(t;\ t_0,x_0)| \leq -\frac{h_3 - 2h_2}{2p}, a.s. for\ all\ x_0 \in \mathbb{R}^d.$ More precisely, if $h_3 > 2h_2$, the trivial solution of Eq. (3) is almost surely exponentially stable.

The following conjectures shall be imposed putting in mind that the conditions of existence and uniqueness of the trivial solutions of the stochastic pantograph differential equation (3) with several deviating arguments hold:

Proposition 3.2:

Let M be a symmetric – definite $d \times d - matrix$. Assume the there exist some positive constants K, θ, ω $such\ that$ $2\theta > 0$, then

$$\left|x^T Mf\left(\left(t, x(t), x(t-r_i), x(q_i t)\right)\right)\right| \leq K|x|^2$$

$$trace\left(G_k^T(x,t)MG_k(x,t)\right)trace\left(D^T(x,t)MD(x,t)\right)trace\left(E^T(x,t)ME(x,t)\right) \leq \theta x^T Mx$$

$$\left|\left(x^T MG_k(x,t)\right)\left(x^T MD(x,t)\right)\left(x^T E(x,t)\right)\right|^2 \geq \omega|x^T Mx|^2, \forall\ t \geq 0\ and\ x \in \mathbb{R}^d.$$

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Following from the idea in Theorem 3.1, it is clear that for sufficiently large noise driving force parameter σ, the trivial solution of Eq. (3) stabilizes itself in an almost sure exponential sense, if σ is replaced by a carefully chosen finite expression.

Proposition 3.3:

There exist a combination of positive constants $Q > 0 \; and \; \delta \geq 0$ such that $\|\eta(t)\| \leq Qe^{\eta t} \; for \; all \; t \geq 0$, where $\eta(t)$ is called a convergent rate function.

$$\int_0^\infty |\eta(t)x(t,x_0)|^p dt \quad < \infty, \; a.s. \tag{12}$$

and hence, the stochastic pantograph delay differential Eq. (11) with multiple deviating arguments, is a.s. exponentially stable.

Proof:

By Proposition (3.2 one sees that $x(t,0) \equiv 0$. it is only necessary to show that

$\sigma = \int_0^\infty |\eta(t)x(t,x_0)|^p dt \quad < \infty,$
$a.s. \; holds \; for \; all \; x_0 \neq 0$. By the

Lemma 3.4:

Assume that (C4.1) is satisfied, the solution of Eq. (3) has the property

$P(x(t,x_0) \neq 0, \; \forall \, t \geq 0) = 1$, provided $x_0 \neq 0$.

Theorem 3.5:

Assume that the Proposition (3.2) and Proposition (3.3) are fulfilled. Then the solutions of the stochastic pantograph differential Eq. (11) possesses the property

property in Lemma 3.4, i.e., $P(x(t,x_0) \neq 0, \; \forall \, t \geq 0) = 1$, provided $x_0 \neq 0, a.s.$, the solution $x(t,x_0) \neq 0, \forall \, t \geq 0$ almost surely. Now assume on the contrary that Inequality (12) is not true, it must be that there exists a given $x_0 \neq 0, a.s.$ for which $P(\Omega^*) > 0, \; for \; some \; \Omega^* \subseteq \Omega$, where

$$\Omega^* = \left\{ \mathrm{w} \in \Omega \colon \int_0^\infty |\eta(t)x(t,x_0)|^p dt \; = \infty, \; a.s. \right\}$$

Applying Ito's formula and Proposition (3.2) it is easily verified that for some given $t \geq 0$

$$log\left(x^T(t)Mx(t)\right) \leq log(x_0^T \, Mx_0) + \frac{2kt}{\sqrt{\lambda_{min(M)}}} + \alpha \int_0^t \left(\int_0^s |\eta(v)x(v)|^p dv\right)^2 ds$$
$$-2\int_0^t \left(\int_0^s |\eta(v)x(v)|^p dv\right)^2 \frac{\left|x^T(s)MG_k(x(s),s),s\right|^2}{\left(x^T(s)Mx(s)\right)^2} \frac{\left|x^T(s)MD(x(s),s),s\right|^2}{\left(x^T(s)Mx(s)\right)^2} \frac{\left|x^T(s)ME(x(s),s),s\right|^2}{\left(x^T(s)Mx(s)\right)^2} ds$$
$$+ \; Q(t) \tag{13}$$

Where, $\sqrt{\lambda_{min(M)}}$ is the smallest Eigen value of the matrix M and Q(t) is the expression

$$Q(t) = 2\int_0^t \left(\int_0^s |\eta(v)x(v)|^p dv\right)^2 \frac{|x^T(s)MG_k(x(s),s),s|^2}{\left(x^T(s)Mx(s)\right)^2} \frac{|x^T(s)MD(x(s),s),s|^2}{\left(x^T(s)Mx(s)\right)^2}$$

$$\frac{|x^T(s)ME(x(s),s),s|^2}{\left(x^T(s)Mx(s)\right)^2}\left(\left[\sum_{k=1}^{m} G_k x(t)dB_k(t)\right] + \int_{|y|<\Delta} D(y)\,x(t)\bar{N}(dt,dy)\right.$$

$$\left. + \int_{|y|\geq\Delta} E(y)\,x(t)N(dt,dy)\right)ds$$

is a continuous Martingale vanishing at t = 0. Assume k = 1, 2, …, it follows from the exponential Martingale inequality that

$$P\left(w: \sup_{0\leq t\leq k}\left[Q(t) - \tfrac{2\omega-\theta}{8\omega}\langle Q(t),\ Q(t)\rangle\right] > \tfrac{8\omega logk}{2\omega-\theta}\right) \leq \tfrac{1}{k^2}$$

Where

$$\langle Q(t), Q(t)\rangle = 4\int_0^t \left(\int_0^s |\eta(v)x(v)|^p dv\right)^2 \frac{|x^T(s)MG_k(x(s),s),s|^2}{\left(x^T(s)Mx(s)\right)^2} \frac{|x^T(s)MD(x(s),s),s|^2}{\left(x^T(s)Mx(s)\right)^2}$$

$\frac{|x^T(s)ME(x(s),s),s|^2}{\left(x^T(s)Mx(s)\right)^2}.$

$$\left(\left[\sum_{k=1}^{m} G_k X(t)dB_k(t)\right] + \int D(y)\,X(t)\bar{N}(dt,dy) + \int E(y)X(t)N(dt,dy)\right)ds$$

Applying Borel- Cantelli Lemma, one gets that given almost all $w \in \Omega$, there exists a random integer $k_1(w)\ such\ that\ \forall\ k\ >\ k_1(w)$,

$$\sup_{0\leq t\leq k}\left[Q(t) - \tfrac{2\omega-\theta}{8\omega}\langle Q(t), Q(t)\rangle\right] \leq \tfrac{8\omega logk}{2\omega-\theta}$$

That is, for $0\ \leq t \leq k$,

$$Q(t)\ \leq\ \frac{8\omega logk}{2\omega-\theta} + \frac{2\omega-\theta}{8\omega}\ \langle Q(t), Q(t)\rangle$$

$$\leq \frac{8\omega logk}{2\omega-\theta} + \frac{2\omega-\theta}{8\omega}\int_0^t\left(\int_0^s|\eta(v)x(v)|^p dv\right)^2 \frac{\left|x^T(s)MD(x(s),s),s\right|^2}{\left(x^T(s)Mx(s)\right)^2}\frac{\left|x^T(s)ME(x(s),s),s\right|^2}{\left(x^T(s)Mx(s)\right)^2}ds \qquad (14)$$

Substituting (14) into (13) and applying Proposition (3.2) one gets

$$log\left(x^T(t)Qx(t)\right) \leq log(x_0^T M x_0) + \frac{2kT}{\lambda_{min(M)}} + \frac{8\omega logk}{2\omega-\theta}$$

$$-\frac{2\omega-\theta}{2}\int_0^t\left(\int_0^s|\eta(v)x(v)|^p dv\right)^2 ds \qquad (15)$$

for all $0 \leq t \leq k, k \geq k_1$ almost surely. By the definition of Ω^*, it is clear that for every $\mathrm{w} \in \Omega^*$, there exists a random integer $\mathrm{k}_2(\mathrm{w})$ such that

$$\int_0^t|\eta(s)x(s)|^p ds \geq \frac{\sqrt{\frac{4k}{\lambda_{min(M)}{}^{+4\delta+8}}}}{2\omega-\theta}, \text{ for all } k \geq k_1 \qquad (16)$$

It then follows from (15) and (16) and almost all $w \in \Omega^*$ that, if $\mathrm{K}-1 \leq \mathrm{t} \leq \mathrm{k}$, $\mathrm{K} \geq \mathrm{k}_1 \vee (\mathrm{k}_2+1)$,

$$\log\left(\mathrm{x}^{\mathrm{T}}(\mathrm{t})\mathrm{Qx}(\mathrm{t})\right) \leq \log(\mathrm{x}_0^{\mathrm{T}}\mathrm{Qx}_0) + \frac{2\mathrm{Kk}}{\lambda_{min(M)}} + \frac{8\omega\mathrm{logk}}{2\omega-\theta}$$

$$-\frac{2\omega-\theta}{2}\int_0^t\left(\int_0^s|\eta(v)x(v)|^p dv\right)^2 ds$$

$$\leq \log(\mathrm{x}_0^{\mathrm{T}}\mathrm{Qx}_0) + \frac{2\mathrm{Kk}}{\lambda_{min(M)}} + \frac{8\omega\mathrm{logk}}{2\omega-\theta} - \left(\frac{2\mathrm{K}}{\lambda_{min(M)}} + 2\delta + 4\right)(\mathrm{K}-1-\mathrm{k}_2)$$

$= \log(\mathrm{x}_0^{\mathrm{T}}\mathrm{Qx}_0) + \frac{2\mathrm{K}(\mathrm{k}_2+1)}{\lambda_{min(M)}} + \frac{8\omega\mathrm{logk}}{2\omega-\theta} - 2(\delta+2)(\mathrm{K}-1-\mathrm{k}_2)$ and as such,

$$\frac{1}{\mathrm{t}}\log\left(\mathrm{x}^{\mathrm{T}}(\mathrm{t})\mathrm{Qx}(\mathrm{t})\right) \leq \frac{1}{\mathrm{k}-1}\log(\mathrm{x}_0^{\mathrm{T}}\mathrm{Qx}_0) + \frac{2\mathrm{K}(\mathrm{k}_2+1)}{\lambda_{min(M)}} + \frac{8\omega\mathrm{logk}}{2\omega-\theta}$$

$-2(\delta+2)(K-1-k_2)$, which then follows that

$\underset{t\to\infty}{LimSup}\frac{1}{t}log\left(\mathrm{x}^{\mathrm{T}}(\mathrm{t})\mathrm{Qx}(\mathrm{t})\right) \leq 2(\delta+2), \ \forall\, t \geq k_3$ and as such,

$|x(t)| \leq \frac{e^{-(\delta+2)t}}{\sqrt{\lambda_{min(M)}}}, \forall\, t \geq k_3$. Now applying Prop (3.3) for almost all w $\in \Omega^*$,

$$\int_0^\infty |\eta(t)x(t)|^p dt \leq \int_0^{k_3} M^p e^{p\delta t}|x(t)|^p\, dt + \int_{k_3}^\infty \frac{M^p e^{p\delta t}}{\left|\lambda_{min(M)}\right|^{p/2}} dt < \infty$$

This is a contradiction, by the definition of Ω^* and hence, the property

$\int_0^\infty |\eta(t)x(t,x_0)|^p dt \quad < \infty,\ a.s.$ holds.

CONCLUSION

The crucial condition which ensures the a.s. exponential stability of the trivial solution of Eq.(11) is $\int_0^\infty |\eta(t)x(t,x_0)|^p dt \quad < \infty$, $a.s.$ In the comparable deterministic pantograph differential Eq. (2), where Levy noise is absent, one sees that for some w $\in \Omega$, the expression $\int_0^\infty |x(t,w)| dt = \infty$ and at that point, the trivial solution of the deterministic Eq.(2.1) can never be stable.

REFERENCES

Negative Coefficients. Transactions of the Nigerian Association of Mathematical Physics, Vol. 5, 27 – 32.

Ahmad, I. and Mukhtar, A. (2015). Stochastic Approach to the Solution of multi-pantograph differential Equation arising in Cell-growth Model. *Applied Mathematics and Computation*, 261,360 – 372.

Alferov, G., Ivanov, G., Efimova, P., Shalay, A. (2018). Stability of Linear systems with Multitask right-hand Member. Stochastic Methods for Estimation and Problem solving in Engineering (Book Chapter), 74 – 112.

Applebaum, D. and Siakali, M. (2009) Asymptotic Stability of Stochastic Differential Equations Driven by Levy Noise. *Journal of Applied Probability*, 46(4), 1116 -1129.

Applebaum, D. (2004). Levy Process: From Probabity to Finance and Quantum Groups. Notices of the AMS, 5(11), 1337.

Agusto, F. B., Bewick, S. and Fagan, W. F. (2017). Mathematical Model of Zika Virus with Vertical Transmission. *Infectious Disease Modeling*, 2(2); 244-267.

Arnold, L. (1974). Stochastic Differential Equations: Theory and Applications. John Wiley & Sons, U. S. A.

Atonuje, A. O. and Ayoola, E. O. (2007). On Noise Contribution to the Oscillatory Behavior of Solutions of Stochastic Delay Differential Equations. *Journal, Institute Mathematics. & Computer Science (Computer Science Series)*. Vol. 18, No. 2, 51 – 59.

Atonuje, A. O. and Ayoola, E. O. (2008). On the Complementary Roles of Noise and Delay in the Oscillatory Behaviour of Stochastic Delay Differential Equations.

Journal of Mathematical Sciences, Vol. 19, No.1, 11 -20.

Atonuje, A. O. and Ezenweani, U. L. (2011). Instability and Stochastic Stabilization in Solutions of Delay Differential Equations with two Delays. *Journal of Mathematical Sciences*, Vol. 23, No. 1, 207 -214.

Atonuje, A. O. and Apanapudor, J. S. (2017). Almost Sure Exponential Stabilization of First order Neutral Delay Differential Equations with Positive and Negative Coefficients. *Transactions of the Nigerian Association of Mathematical Physics*, Vol. 5, 27 – 32.

Atonuje, A. O. (2010). Analysis of Noise Generated Oscillation of Solutions of Stochastic Delay Differential Equations with Negative and Positive Coefficients. *Journal of the Institute of Mathematics and Computer Science. (Mathematics Series)*, 23(3);87-95.

Atonuje, A. O. and Tsetimi, J. (2016). Suppressive Influence of Time-Space White Noise on the Explosion of Solutions of Stochastic Fokker-Planck Delay Differential Equations. *Journal of Mathematics and System Science,* 6; 284-290.

Baker, C. T. H. and Buckwar, E. (2000). Numerical Analysis of Explicit One-step Methods for Stochastic Differential Equations. London Mathematical Soceity *Journal of Computational Mathematics,* 3 (2000), 315 – 335.

Ezzinbi, M. and Kad, J. (2006). Blow-up Results for some Nonlinear Delay Differential Equations. Positivity, 10, 329-341.

Florescu, I. (2014). Probability Theory and Stochastic Processes. John Wiley & Sons, Engle Cliff Wood, New Jersey.

Fristedt, B. E. and Gray, L. F. (2013). A Model Approach to Probability Theory. Springer Science and Business Media, 580.

Gao, Z., Yang, l. and Luo, Z. (2013). Stability of the Solutions for Nonlinear Fractional Differential Equations with Delays and integral Boundary Conditions. *Advances in Differential Equations*, 43, 1 – 8.

Gopalsamy, K. (1987). Oscillatory Properties of System of First Order Linear Delay Differential Inequalities. *Pacific Journal of Mathematics*, 128(2), 299-305.

Goriety, A. and Hyde, C. (1998). Finite-Time Blow-up in Dynamic Systems. Phy. Letters, 250, 311-318.

Hahn, W. (1967). Stability of Motion. Springer.

Jordan, P. M. (2008). A Note on Burgers with Time Delay: Instability via Finite Blow up. Phy. Letters, 372(42), 6363-6367.

Jordan R. and Turkington, B. (2001). Statistical equilibrium Theories for Nonlinear Schrodinger Equation. Advances in Wave Interaction and Turbulence Contemporary Mathematics, 283; 27 – 39.

Kloeden, P. E. and Platen, E. (1992). Numerical solutions of Stochastic Differential Equations. Springer-Verlag, New York.

Mao, X. (1997). Stochastic Differential Equations and their Applications. Sawston Cambridge: Horwood Publishing Limited.

Rihan, F. A. (2018). Application of Delay Differential Equations in Biological Systems. Complexity, 2018, DOI: http/doi.org/10. 1155/2018/45843889.

Rubab, S. and Ahmad, J. (2016). Analytic Solution of Multi-pantograph Delay Differential Equations via Sumudu Decomposition Method. *Mathematical Theory and Modeling*, 6(6); 92-99.

Sezer, M., Yalcinba, S. and Sahin, N. (2008). Approximate Solution of Multi-pantograph Equation with Variable Coefficients. *Journal of Computational and Applied Mathematics*, 214; 406-416.

Talor, A. B. and Ockendon, J. R. (1971). The Dynamics of a Current Collection System for an Electric Locomotive. Proceedings of the Royal Society of London, 322; 447-464.

Unser, M. (2020). A Note on BIBO Stability. IEEE Translations and Signal processing, 68, 5904 – 5913.

Wu, F., Mao, X. Szpruch, L. (2010). Almost Sure Exponential Stability of numerical Solutions for Stochastic Delay Differential Equations. Numer. Math (2010), 115, 681 -697.

Zhang, W., Song, M. H. and Liu, M. Z. (2019). Almost Sure Exponential Stability of Stochastic Differential Delay Equations. Filomat, 33 (3), 789 -814. Published by Faculty of Sciences and Mathematics, University of Nis, Serbia.

Zhu, Y., Zhu, M. and Li, J. (2017). Almost Sure Exponential Stability for Time-Changed Stochastic Differential Equations. *Journal of Nonlinear Sciences and Applications*, 10 (2017), 5985 – 5998.