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Interpolation-Based Iterative Methods for Nonlinear Problems

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Abstract
Nonlinear problems pose significant challenges in the development of iterative solvers due to costly derivative evaluations, nonexistent or non-smooth derivatives, and singularities. To address these obstacles, this study presents a novel class of Newton-Interpolation-Based Iterative methods (NIBM) that circumvent derivative evaluations for nonlinear problems. A comprehensive convergence analysis framework is established to systematically investigate the convergence rates of NIBM. Numerical experiments demonstrate the efficacy and robustness of NIBM methods in solving nonlinear equations, highlighting their potential for applications in scientific and engineering disciplines. The results underscore the efficiency and reliability of NIBM methods.
Keywords: Newton Interpolation, Nonlinear equations, Iterative methods, Convergence analysis

Introduction

Nonlinear problems, represented by the equation $G(t) = 0$, are prevalent in various scientific and engineering disciplines, including fluid dynamics, structural mechanics, optimization, and control theory (Gdawiec *et al.*, 2019; Soheili *et al.*, 2019; Behl, 2022). These problems often exhibit complex behaviors, characterized by singularities and non-smooth transitions, posing significant challenges in the development of efficient and robust iterative methods. Traditional iterative techniques, which rely on function derivative evaluations, may fail to capture the underlying dynamics, particularly when derivatives are expensive, inaccurate, or impossible to evaluate due to singularities or non-smooth transitions.

The primary obstacle in solving nonlinear problems is the evaluation of derivatives, which can be a significant bottleneck. The classical Newton method (Kung and Traub, 1974) and the traditional Newton-based methods, such as those developed in (Padilla *et al.*, 2022; Ogbereyivwe and Emunefe, 2022; Ogberevivwe *et al.*, 2023), which rely heavily on derivative evaluations, may exhibit slow convergence, divergence, or even failure to converge in such scenarios. To overcome these challenges, researchers have developed alternative methods using divided differences, interpolation polynomials, and weight functions (Soleymani, 2010; Argyros *et al.*, 2017; Bahgat, 2021; Ogberevivwe *et al.*, 2025) and some reference therein. However, these methods often suffer from limitations, such as slow convergence rates, high computational costs, or restrictive assumptions on the problem structure.

In recent years, interpolation-based iterative methods have gained significant attention due to their ability to bypass derivative evaluations and accommodate non-smooth transitions, see (Qureshi, *et al.*, 2017; Ide, 2023). These methods rely on constructing a surrogate model of the nonlinear function derivative using interpolation techniques. However, the development of efficient and robust interpolation-based methods remains

an active area of research, with many challenges still to be addressed.

This paper presents a novel class of Newton-Interpolation-Based Iterative methods (NIBM), which synergistically combine the strengths of Newton-based methods and interpolation-based techniques. The proposed NIBM bypass nonlinear problem derivative evaluations while leveraging the efficiency and robustness of Newton-based methods. A comprehensive convergence analysis framework is established to investigate the convergence rates of NIBM, providing valuable insights into their performance.

The remainder of this paper is organized as follows. Section 2 presents the mathematical formulation of the nonlinear problem and the proposed NIBM. Section 3 provides a comprehensive implementation of the developed NIBM and compared methods, while Section 4 demonstrates their efficacy through numerical experiments. Finally, Section 5 concludes the paper, highlighting the potential of NIBM for applications in scientific and engineering fields.

Materials and Methods

We begin this section by delineating the fundamental conceptual framework and definitions that undergird this work, so as to provide a rigorous mathematical foundation for subsequent analysis.

Definition 1: Given a dataset comprising $(i+1)$ - distinct points $s_0, s_1, \cdots, s_i$, the interpolating polynomial of degree i, denoted as $\Phi_i(s)$, can be expressed mathematically as:

$$\Phi_i(s) = G[s_0] + \sum_{i=0}^{i-1} \Lambda_i(t) G[s_0, s_1, \cdots, s_i], \quad (1)$$

where $\Lambda_i(t)$ represents a coefficient function given as:

$$\Lambda_i(t) = \Pi(s - s_i), \quad (2)$$

and

$$G[s_0, s_1, \cdots, s_i] = \frac{[s_0, s_1, \cdots, s_i] - [s_0, s_1, \cdots, s_{i-1}]}{s_i - s_0}. \quad (3)$$

Definition 2: The residual error of an iterative method is defined as:

$$e_{i+1} = \eta e_i^{\rho} + O\left(e_i^{\rho+1}\right), \quad (4)$$

where η denotes the error constant, ρ represents the theoretical convergence order, e_i signifies the error at the ith iteration, and $O\left(e_i^{\rho+1}\right)$ represents the higher-order terms, asymptotically bounded (Ogbereyivwe *et. al.*, 2024).

Definition 3: The computational convergence order of an iterative scheme, denoted as ρ_c , can be computationally quantified using the following metric:

$$\rho_c = \frac{|\log \gamma_1|}{|\log \gamma_2|} \quad (5)$$

where $\gamma_1 = \frac{e_{i+1}}{e_i}$ and $\gamma_2 = \frac{e_i}{e_{i-1}}$, for $i \geq 3$ (Petkovic *et al.*, 2012).

Subsequent to the establishment of the underlying mathematical framework, we proceed to derive a rigorous methodological formulation, thereby integrating the theoretical foundations with computational paradigms to facilitate a robust numerical solution.

Consider the traditional one-step Newton method given as:

$$t_{i+1} = t_i - \frac{G(t_i)}{G'(t_i)} \quad , \quad i = 0,1,2,\cdots \tag{6}$$

To annihilate function derivative $G'(t_i)$ in (6), the points $\omega_1 = t_i + \theta(G(t_i))^5$ and $\omega_2 = t_i - \theta\big(G(t_i)\big)^5$, $\theta \in R$ are utilised to approximate $G'(t_i)$ using the Newton divided difference interpolation polynomial $\Phi_1(s)$ at some points, such that $\Phi_1'(s) \approx G'(t)$. It is important to note that the choice of the exponent 5 in $(G(t_i))^5$, is to optimize precision of the method.

Now from (1), we have

$$\Phi_1(s) = G(\omega_1) + (s - \omega_1)G[\omega_1, \omega_2]. \tag{7}$$

and,

$$\Phi_1'(s) = G[\omega_1, \omega_2]. \tag{8}$$

Setting $s = t$, we have:

$$\Phi_1'(t) = G[\omega_1, \omega_2] \approx G'(t). \tag{9}$$

Consequently, the modified form of the iterative method in Equation (1) denoted as $NIBM2$ is proposed as:

$$t_{i+1} = t_i - \frac{2\theta\big(G(t_i)\big)^6}{G(\omega_2) - G(\omega_1)}. \tag{10}$$

Consider the double step method given as:

$$\begin{cases} y_i = t_i - \dfrac{2\theta\big(G(t_i)\big)^6}{G(\omega_2) - G(\omega_1)} \\ t_{i+1} = \quad y_i - \dfrac{G(y_i)}{G'(y_i)}. \end{cases} \tag{11}$$

If we use the already available data ω_1, ω_2 and $\omega_3 = y_i - \theta\big(G(y_i)\big)^5$ to obtain a 2nd - order Interpolation polynomial $\Phi_2(s)$ such that $\Phi_2(s) \approx G'(y_i)$ at some points, then:

$$\Phi_2(s) = G(\omega_1) + (s - \omega_1)G[\omega_1, \omega_2] + (s - \omega_1)(s - \omega_2)G[\omega_1, \omega_2, \omega_3], \tag{12}$$

and

$$\Phi_2'(s) = (2s - \omega_1 - \omega_2)G[\omega_1, \omega_2, \omega_3] + G[\omega_1, \omega_2]. \tag{13}$$

Upon setting $s = y$ in Equation (13), we have:

$$\Phi_2'(y) = (2y - \omega_1 - \omega_2)G[\omega_1, \omega_2, \omega_3] + G[\omega_1, \omega_2]. \tag{14}$$

Based on Equation (14), we propose the iterative method $NIBM4$ as:

$$\begin{cases} y_i = t_i - \dfrac{2\theta\big(G(t_i)\big)^6}{G(\omega) - G(\mu)}, \\ t_{i+1} = \quad y_i - \dfrac{G(y_i)}{(2y - \omega_1 - \omega_2)G[\omega_1, \omega_2, \omega_3] + G[\omega_1, \omega_2]}. \end{cases} \tag{15}$$

Next, we conduct the convergence analysis of the proposed methods $NIBM2$ and $NIBM4$. Since the convergence of method $NIBM4$ subsumes the convergence of $NIBM2$, we prove the convergence of $NIBM4$.

Theorem 1. Let $G : \sigma \subset R \to R$ be a single variable function that is at least two times differentiable in the neighbourhood of $t_* \subset \sigma$, where t_* is the simple solution of the function G. For any chosen t_0 in the neighbourhood of t_* the iterative methods $NIBM2$ and $NIBM4$ will produce a sequence converging to t_* with second and quartic convergence rates respectively.

Proof. Consider the Taylor's expansion of $G(t)$ about t_* and setting $t = t_i$, we have:

$$G(t) = G'(t_*)\left[e_i + \sum_{j=2}^{4} \delta_j e_i^j + O\left(e_i^5\right)\right], \quad i = 1,2,\ldots \tag{16}$$

whereas $\delta_j = \frac{G^{(j)}(t_*)}{j!G'(t_*)}$, $j = 2,3,\ldots$

Also, the Taylor's expansions of $G(\omega_1)$ and $G(\omega_2)$ about t_* is obtained as

$$G(\omega_1) = G'(t_*)\left[e_i + \delta_2 e_i^2 + \delta_3 e_i^3 + \delta_4 e_i^4 + O\left(e_i^5\right)\right], \tag{17}$$

and

$$G(\omega_2) = G'(t_*)\left[e_i + \delta_2 e_i^2 + \delta_3 e_i^3 + \delta_4 e_i^4 + O\left(e_i^5\right)\right], \tag{18}$$

respectively.

Using Equations (17) and (18), we have obtained the expansion for the divided difference $G[\omega_1, \omega_2]$ as:

$$G[\omega_1, \omega_2] = 1 + 2\delta_2 e_i + 3\delta_3 e_i^2 + 4\delta_4 e_i^3 + O\left(e_i^5\right). \tag{19}$$

By the application of Equations (16-18), the expansion of the first iterative step of Equation (15) is obtained as:

$$y = t_i - \frac{2\theta\left(G(t_i)\right)^6}{G(\omega) - G(\mu)}$$
$$= \delta_2 e_i^2 + (-2\delta_2^2 + 2\delta_3)e_i^3 + (4\delta_2^3 - 7\delta_2\delta_3 + 3\delta_4)e_i^4 + O\left(e_i^5\right). \tag{20}$$

Using Equation (20), the Taylor's expansion of y_i about zero is obtained as:

$$G(y) = G'(t_*)\left[\delta_2 e_i^2 + (-2\delta_2^2 + 2\delta_3)e_i^3 + (5\delta_2^3 - 7\delta_2\delta_3 + 3\delta_4)e_i^4 + O\left(e_i^5\right)\right]. \tag{21}$$

Applying Equations (17-21), the series expansion for the divided difference $G[\omega_1, \omega_2, \omega_3]$ in Equation (15) is obtained as:

$$G[\omega_1, \omega_2, \omega_3] = \delta_2 + 2\delta_3 e_i + (\delta_2\delta_3 + 3\delta_4)e_i^2 + O(e_i^3). \tag{22}$$

From Equations (19), (20) and (22) we have:

$$(2y - \omega_1 - \omega_2)G[\omega_1, \omega_2, \omega_3] + G[\omega_1, \omega_2] = 1 + (2\delta_2^2 - \delta_3)e_i^2 - 2(2\delta_2^3 - 3\delta_2\delta_3 + \delta_4)e_i^3 + O(e_i^4) \tag{23}$$

Consequently, from the expansions in Equations (21) and (23), we have:

$$\frac{G(y_i)}{(2y - \omega_1 - \omega_2)G[\omega_1, \omega_2, \omega_3] + G[\omega_1, \omega_2]} = \delta_2\delta_2^2 + (-2\delta_2^2 + \delta_3)e_i^3 + 3(\delta_2^3 - 2\delta_2\delta_3 + \delta_4)e_i^4 + O\left(e_i^5\right). \tag{24}$$

By substituting Equation (24) into the second step of Equation (15) and after some simplifications, we have:

$$z_{i+1} = t_* + (\delta_2^3 - \delta_2\delta_3)e_i^4 + O\left(e_i^5\right). \tag{25}$$

This ends the proof.

Remark 1. A careful examination of the proof of Theorem 1, coupled with the observation that Method $NIBM4$ subsumes Method $NIBM2$, enables a direct comparison with the general form of residual equation in Definition 2 (see Equation (4)) and the residual equations derived in Equations (20) and (25). This comparison yields a clear convergence hierarchy, wherein Methods $NIBM4$ and $NIBM2$ achieve convergence orders of 4 and 2 respectively.

Results

A comprehensive assessment of the developed methods $NIBM2$ and $NIBM4$, through their application to range of nonlinear equations drawn from the literature, is presented in this section. Their

computational efficiency is bench-marked against established methods of comparable convergence order, including the Kung-Traub (1974) method denoted as (TM2) with convergence order 2 and given as

$$t_{i+1} = t_i - \frac{G(t_i)}{G[t,\beta]}, \quad \beta = t_i + G(t_i), \tag{26}$$

and the following order 4 methods:
Kung-Traub (1974) (TM4):

$$\begin{cases} y_i = t_i - \dfrac{G(t_i)}{G[t,\beta]}, \\ t_{i+1} = y_i - \dfrac{G(y_i)G(\beta)}{\big(G(\beta) - G(y)\big)G[t,y]}, \end{cases} \tag{27}$$

Zheng *et al.*, (2011) (ZM4):

$$\begin{cases} y_i = t_i - \dfrac{G(t_i)}{G[t,\beta]}, \\ t_{i+1} = y_i - \dfrac{G(y_i)}{G[t,y] + (y_i - t_i)G[t,\beta,y]}, \end{cases} \tag{28}$$

Ogbereyivwe *et al.*, (2025) (OM4):

$$\begin{cases} \mathrm{y_i} = \mathrm{t_i} - \dfrac{\mathrm{G(t_i)}}{\mathrm{G[t,\ \mu]}}, \qquad \mu = \mathrm{t_i} + \alpha\big(G(\mathrm{t_i})\big)^2 \\ t_{i+1} = \mathrm{y_n} - \left[1 - 2\left(\dfrac{\mathrm{G(y_i)}}{\mathrm{G(t_i)}} - 1\right) - 2\left(\dfrac{\mathrm{G(y_i)}}{\mathrm{G(t_i)}} - 1\right)^2\right]\dfrac{\mathrm{G(y_i)}}{\mathrm{G[t,\ \mu]}}, \end{cases} \tag{29}$$

Bahgat (2021) (BM4):

$$\begin{cases} y_i = t_i - \dfrac{2\theta G^2(t_i)}{G\big(t_i + \theta G(t_i)\big) - G\big(t_i - \theta G(t_i)\big)}, \\ t_{i+1} = t_i - \left(\dfrac{G^2(t_i) - G(t_i)G(y_i) + G^2(y_i)}{G^2(t_i) - 2G(t_i)G(y_i) + G^2(y_i)}\right)\dfrac{2\theta G^2(t_i)}{G\big(t_i + \theta G(t_i)\big) - G\big(t_i - \theta G(t_i)\big)}. \end{cases} \tag{30}$$

A specialized computer program was created in the Maple environment to implement all the methods, with computations performed to a high degree of precision (2000 digits) to mitigate truncation errors. Convergence was ensured through the application of a stringent stopping criterion, $|G(t_i)| < 10^{-200}$. The evaluation metrics employed include the number of iterations required for convergence (iter), residual error $|G(t_{i+1})|$, computational convergence order (ρ_c) and residual error fall. The specific test problems used in this evaluation are taken from both academic and applied fields.

Problem 1: To determine the solution $t_* = 2$ of $G_1(t) = (t-1)^2 - 1 = 0$.

Problem 2: Find the solution $t_* = 0.85705677\cdots$. of $G_2(t) = -.5 - cos(x) + tan(x) = 0$.

Problem 3: Consider determining the solution $t_* = 1$, of the non-smooth function $G_3(t) = 0$ defined on $\Omega = [0, 1]$ given as:

$$G_3(t) = \begin{cases} t^3 \ln t^2 - t^2 + t^4, & t \neq 0, \\ 0, & t = 0. \end{cases}$$

The computational results obtained when the developed and compared methods were used to solved problems $G_i(t),\ i = 1,2,3$ are presented in Table 1.

Table 1: Methods computational results for Problems $G_1(t)$, $G_2(t)$ and $G_3(t)$.

$G_i(t)$	Methods	t_0	$\lvert G(t_1)\rvert$	$\lvert G(t_2)\rvert$	$\lvert G(t_3)\rvert$	$\lvert G(t_{i+1})\rvert$	$iter$	ρ_c
$G_1(t)$	TM2	0.2	1.7269E-00	1.1450E-00	6.5554E-01	9.1160E-259	13	1.9923
	NBIM2		5.1472E-01	5.6101E-02	9.8750E-04	3.7718E-223	9	2.0090
	TM4		3.8370E-01	6.4370E-03	1.6570E-09	3.3668E-563	6	3.9929
	OM4		1.8839E-00	1.2402E-00	4.4510E-01	2.8197E-633	8	4.0001
	BM4		7.4685E-02	1.6083E-06	4.1305E-25	6.4336E-394	5	4.0101
	ZM4		2.9055E-01	1.3452E-02	1.2889E-12	3.8312E-770	6	3.9896
	NIBM4		4.3720E-02	8.2416E-08	1.1392E-30	7.3818E-488	5	4.0000
$G_2(t)$	TM2	1.1	10.887E-00	12.258E-00	2.2774E-01	7.8817E-266	12	2.0000
	NIBM 2		2.2227E-01	1.4208E-02	6.3500E-05	8.1387E-301	9	2.0012
	TM4		6.3182E-00	6.7951E-00	3.7343E-01	4.7909E-620	8	4.0000
	OM4		1.5706E-00	1.3246E-01	9.2965E-05	2.0434E-263	6	3.9848
	BM4		1.1303E-02	3.7280E-10	4.4555E-40	1.5752E-638	5	3.9875
	ZM4		1.2976E-00	3.2454E-01	1.0461E-03	4.2659E-209	6	4.0190
	NIBM 4		5.6361E-03	8.7679E-12	5.1245E-47	1.1084E-751	5	4.0000
$G_3(t)$	TM2	0.8	Failed to converge					
	NIBM 2		1.9570E-00	4.9596E-01	8.5208E-02	1.8496E-306	11	2.0000
	TM4		4.5994E-00	9.9467E-02	1.1137E-02	9.2306E-202	>60	1.0100
	OM4		4.5501E-00	4.5396E-00	4.5284E-00	1.0860E-724	>60	1.0100
	BM4		1.3628E-00	5.8522E-02	8.1204E-06	1.0085E-326	6	3.9756
	ZM4		3.0773E-01	4.0830E-02	3.3202E-03	7.4258E-202	>60	1.0100
	NIBM4		2.7776E-01	7.0731E-04	9.9398E-14	9.1427E-211	5	3.9811

To elucidate the comparative dynamics of error convergence, a visual representation of the residual error decay for the developed and compared methods is provided in Figures 1 - 4, wherein the error decay for Problems $G_i(t),\ i = 2,3,4$ are juxtaposed, thereby facilitating a quantitative assessment of the method's relative efficiencies in achieving convergent accuracy.

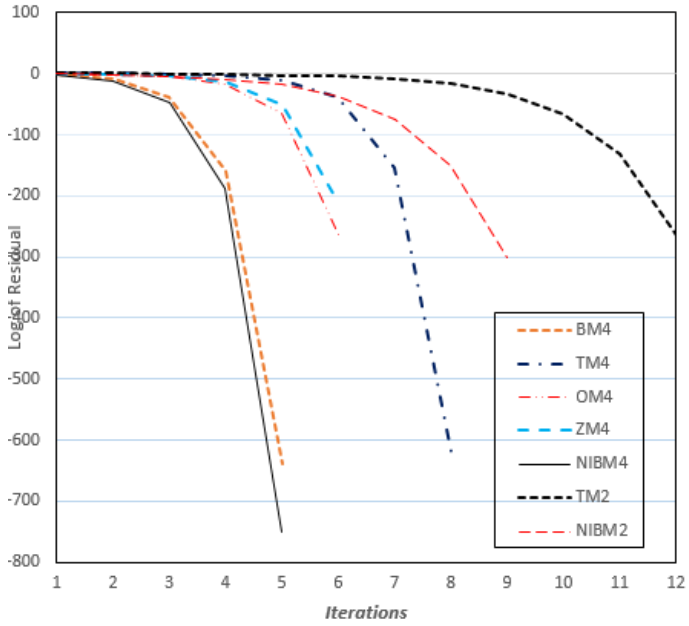

Figure 1: Methods Residual fall comparison for Problem $G_2(t)$

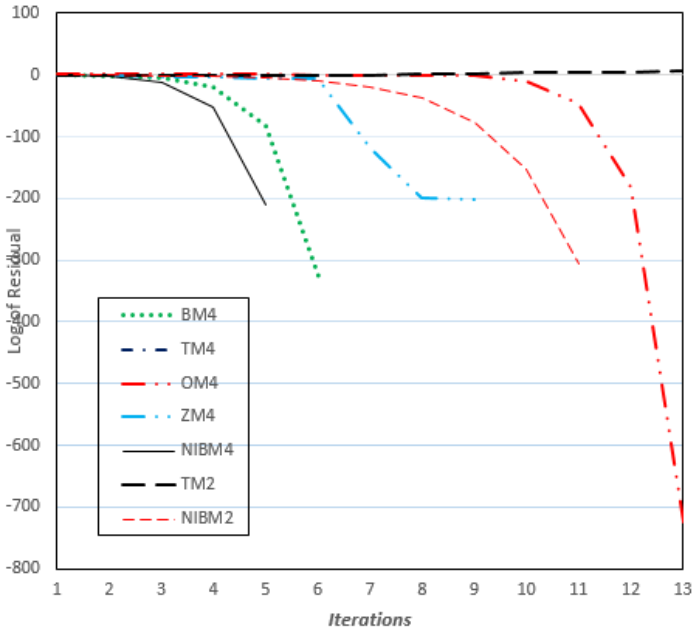

Figure 2: Methods Residual fall comparison for Problem $G_3(t)$

Next, we will apply the developed and compared methods to a selection of real-world engineering problems, thereby assessing their practical efficiency and performance in solving complex industry-relevant challenges.

Problem 4: Estimating Vertical Stress in an Elastic Continuum.

Consider a scenario where a strip base exerts a uniform pressure p on an elastic continuum. Boussinesq's formula in Griffiths and Smith

(2011) describes the resulting vertical stress τ at a point beneath the strip base:

$$\tau = \frac{p}{\pi} + \sin(t)\, cost(t). \qquad (31)$$

A researcher aims to find the value of t at which the vertical stress τ equals 25% of the footing stress p. Assuming an initial estimate of $t = 0.9$, the equation can be rewritten as:

$$G_4(t) = \frac{t + \sin(t)\, cost(t)}{\pi} - 0.25. \qquad (32)$$

The goal is to find the root of $G_4(t) = 0$, starting from the initial estimate $t = 0.9$. This problem will be solved using various numerical methods, including a proposed methods, to compare their performance.

Table 2: Methods computational results for Problems $G_4(t)$

Methods	t_0	$\|G(t_1)\|$	$\|G(t_2)\|$	$\|G(t_3)\|$	$\|G(t_{i+1})\|$	$iter$	ρ_c
TM2		3.2130E-01	1.0717E-02	1.5632E-04	9.0229E-238	9	2.0000
NIBM2		1.7251E-01	9.9349E-03	9.7439E-05	5.6020E-269	2	1.9926
TM4		1.5482E-01	4.7691E-04	2.3238E-13	1.5482E-795	6	3.9950
OM4	0.9	9.1985E-01	2.5063E-01	29035 E-00	2.8517E-698	8	3.9770
BM		5.5554E-02	1.0557E-05	2.3887E-20	2.9543E-313	5	3.9620
ZM		6.2898E-02	2.8007E-05	1.9559E-18	1.4900E-281	5	3.9577
NIBM4		2.3725E-02	3.5095E-07	2.0530E-26	4.5235E-411	5	3.9903

Problem 5: Surface water flow system.

Designing standard and ecological water systems poses a significant challenge: accurately modeling the flow of water through open channels, such as trenches, ditches, drains, and sewers. The flow rate of a stream is defined as the volume of water passing a specific point over a set period. However, when channels become clogged, predicting water flow becomes even more complex. In this scenario, Manning's equation plays a crucial role in determining the flow of water in an open channel under steady conditions. This equation is expressed as:

$$G(t) = \sqrt[3]{\left(\frac{tW}{2t + W}\right)^2}\left(\frac{\sqrt{m}}{n} tW\right) - Q \qquad (33)$$

To apply this equation, we must first determine the depth of water (t) in the channel. Assuming a flow rate (Q) of 14.15 m^3/s, channel width (W) of 4.572 m, Manning's roughness coefficient (n) of 0.017, and a minor loss coefficient (m) of 0.0015, we can estimate the initial water depth (t_0) to be 18.5 m. Table 5 presents the mathematical results obtained using various methods.

Table 3: Methods computational results for Problems $G_5(t)$

Methods	t_0	$\|G(t_1)\|$	$\|G(t_2)\|$	$\|G(t_3)\|$	$\|G(t_{i+1})\|$	$iter$	ρ_c
TM2	18.5	6.8008E-02	6.0700E-07	4.8862E-017	7.2853E-320	7	2.0253
NIBM2		5.2981E-02	2.7776E-07	7.5816E-018	9.1728E-335	7	2.0181
TM4		3.0275E-04	9.6566E-24	9.9964E-102	1.1479E-413	4	4.0490
OM4		4.3386E-05	9.7037E-27	2.4286E-113	9.5289E-460	4	4.0708
BM		5.9781E-05	8.0619E-27	2.6666E-116	3.1919E-464	4	4.0702
ZM		Failed to converge					
NIBM4		6.9348E-05	1.4578E-26	2.8464E-113	4.1379E-460	4	4.0708

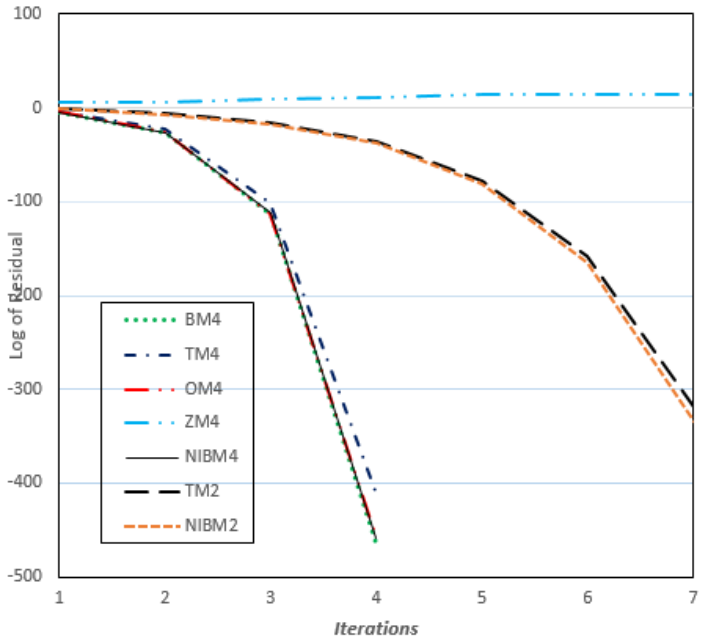

Figure 3: Methods Residual fall comparison for Problem $G_5(t)$

Problem 6: Utilizing a Fractional Approach for Ammonia Synthesis from Nitrogen and Hydrogen (Solaiman and Ishak, 2021).

In chemical processes, the fractional conversion metric represents the proportion of nitrogen and hydrogen converted into ammonia. This metric enables the tracking of reaction progress and the identification of optimal conditions for maximum yield. The fractional conversion calculates the percentage of ammonia produced relative to the total nitrogen and hydrogen input into the reaction system. This ratio serves as a key indicator of reaction efficiency, guiding condition adjustments to optimize ammonia production. Mathematically, the fractional conversion is expressed as the ratio of ammonia produced to the total nitrogen and hydrogen input. Alternatively, it can be calculated by dividing the total nitrogen and hydrogen input by the amount of ammonia produced, yielding a ratio that indicates the amount of ammonia (t) produced from nitrogen and hydrogen. Fractional conversions can be expressed in decimal or percentage forms and are useful for monitoring reaction progress and predicting completion. In this specific problem, temperature and pressure values are set at 500°C and 250 atm, respectively. The

problem is formulated as a non-linear equation:

$$G_6(t) = -7.79075t^3 + 14.7445t^2 + 2.511t - 1.674 = 0 \quad (34)$$

with real roots $t = 0.384094\ldots$ and $t = 0.27776\ldots$. Computational results for an initial value of $t = 0.1$ are presented in Table 4, facilitating performance comparisons among various methods.

Table 4: Methods computational results for Problems $G_6(t)$

Methods	t_0	$\|G(t_1)\|$	$\|G(t_2)\|$	$\|G(t_3)\|$	$\|G(t_{i+1})\|$	$iter$	ρ_c
TM2	0.1	1.5174E-00	1.6514E-00	1.7302E-00	2.4490E-264	25	1.9823
NIBM2		6.4493E-01	3.1993E-02	1.0842E-04	2.2152E-315	9	2.0064
TM4		8.4678E-01	9.6130E-03	2.8496E-09	3.4260E-557	6	3.9786
OM4		1.0020E-01	1.3458E-05	3.1156E-21	2.6231E-432	5	3.9762
BM		1.1792E-01	5.8443E-07	4.0975E-28	3.3700E-451	5	3.9912
ZM		2.3924E-00	1.5647E-00	7.2812E-00	6.3243E-313	8	4.0128
NIBM4		5.5602E-02	1.9378E-08	3.1774E-34	6.2708E-550	5	4.0150

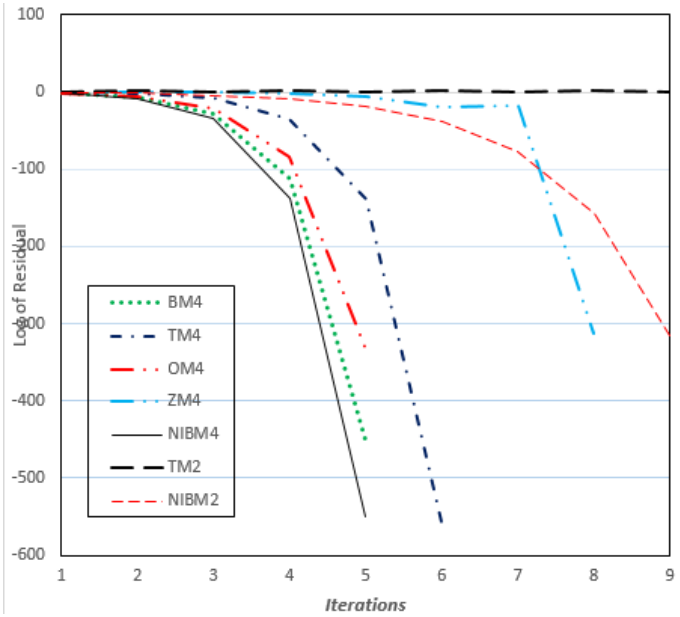

Figure 4: Methods Residual fall comparison for Problem $G_6(t)$.

Discussion

A comprehensive analysis of the numerical results presented in Tables 1-4 reveals the superior accuracy of the proposed Newton-Interpolation Based Iterative Methods ($NIBM2$ and $NIBM4$) compared to existing methods. Notably from Tables 1-4, the proposed methods consistently demonstrate approximately one or two-digit accuracy in the initial iteration, whereas competing methods often require additional iterations to achieve comparable accuracy. This disparity in initial accuracy suggests that the proposed Newton-Interpolation Based Iterative Methods possess a more effective starting mechanism, enabling them to converge more rapidly towards the solution. Furthermore, the developed $NIBM2$ and $NIBM4$ methods

exhibit a consistent computational order of convergence, aligning with the theoretical convergence order derived in the preceding section. In contrast, certain competing methods, such as $TM2, ZM4$, and $OM4$, failed to solve specific problems ($G_3(t), G_5(t)$ and $G_6(t)$, respectively). This highlights the robustness of the proposed NIB methods, which successfully addressed all test problems.

A visual numerical simulations of the methods' error convergence is provided in Figures 1-4, illustrating the residual fall or error decay as the number of iterations increases. The proposed methods ($NIBM2$ and $NIBM4$) demonstrate a more rapid decline in error, indicating their superior accuracy and efficacy. Specifically, Figure 1 illustrates that NIBM4 exhibits a faster asymptotic convergence rate compared to existing fourth-order methods (TM4, BM4, OM4 and ZM4), with error residual diminishing rapidly as iteration counts increase. Similarly, NIBM2 outperforms the second-order method TM2, showcasing enhanced error decay properties. This trend is consistently observed across Figures 2-4, underscoring the robustness and accuracy of the proposed numerical framework.

Collectively, these findings suggest that the proposed methods ($NIBM2$ and $NIBM4$) not only exhibit improved initial accuracy but also maintain their precision throughout the iterative process, making them a reliable choice for solving complex numerical problems.

Conclusion

In conclusion, this study has successfully developed and validated a novel class of Newton-Interpolation-Based Iterative Methods (NIBM2 and NIBM4) for solving nonlinear problems. The comprehensive numerical analysis and experiments have unequivocally demonstrated the superior accuracy, robustness, and stability of the proposed NIB methods compared to existing methods. The rapid convergence, improved initial accuracy, and consistent computational order of convergence of the developed methods make them a reliable and efficient choice for solving complex numerical problems. The findings of this study contribute significantly to the development of iterative methods for nonlinear problems, offering a promising approach for addressing challenging problems in scientific and engineering disciplines. Future research directions may include further extensions and applications of the proposed methods to other complex problems and also address their stability, incorporating stability polynomials and rigorous analysis to ensure their robustness.

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Authors Contributions

The respective contributions of the authors are as follows: O. Ogbereyivwe conceptualized the methodology, conducted theoretical analyses, and provided overall supervision. K. Emumejaye contributed to the development of applied problems, researched the background and undertook proofreading and editing responsibilities, S. O. E. Inonoje and B. E. Idisi performed numerical simulations, results discussion and prepared the manuscript.

Conflict of Interest Declaration

There exist no conflict among the authors regarding the publication of this manuscript.

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