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A REVIEW OF NUMERICAL METHODS FOR STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract
This paper focused on the review of numerical methods for Stochastic Differential Equations (SDEs). The basic theory of stochastic Taylor expansion and the two main types of convergence; the strong and weak convergence are discussed. The strong convergence uses the concept of the absolute error, which is the expectation of the absolute value of the difference between the numerical approximation and the true solution at a specific time. While in the weak convergence, we compute moments of the solution and not to approximate individual paths of the solution. The Euler-Maruyama, Milstein and Runge-Kutta methods were reviewed. Numerical illustrations were carried out and compared. The Runge-Kutta scheme gave the best result.

Keywords: Stochastic Taylor Expansion, Strong and Weak Approximation, Euler-Maruyama and Milstein Method, Runge-Kutta iterative scheme.

Introduction

The motivation for the study is the desire to understand better the numerical solution of a non-linear Stochastic Differential Equations (SDEs). A stochastic differential equation (SDE) is a differential equation in which one or more of the terms is a stochastic process, resulting in a solution which is also a stochastic process. Typically, SDEs contain a variable which represents random white noise calculated as the derivative of Brownian motion or the Wiener process. However, other types of random behaviour are possible, such as jump processes.

The most common form of SDEs in the literature is an ordinary differential equation with the right hand side perturbed by a term dependent on a white noise variable. In most cases, SDEs are understood as continuous time limit of the corresponding stochastic difference equations. This understanding of SDEs is ambiguous and must be complemented by an "interpretation". The most famous interpretations are provided by Itô and Stratonovich calculi, with the former being most frequently used in mathematics and quantitative finance and furnishes a very important tool of constructing diffusion processes. (Akinbo, Faniran, and Ayoola, 2015).

As more realistic mathematical models become required to take into account random effects and influences in real world systems stochastic differential equations (SDEs) have become essential in the accurate description of such situations. Since SDEs rarely have explicit solutions, approximate numerical methods are vital in order to make their implementation viable. Due to features of the stochastic calculus the numerical analysis of SDE's differs in some key areas from the already well-developed area of the numerical analysis of ordinary differential equations, but much of this theory can be extended to the stochastic case also (Burrage, Burrage, Tian, 2004)

Consider a one dimensional stochastic differential equation (SDE) of the form below:

$$dX_t = \emptyset(X_t)dt + \Phi(X_t)dB_t \quad (1)$$

for $t \in [0,T]$, with initial value $\chi_0 \epsilon R$. The stochastic process $X = \{X_t, 0 \leq t \leq T\}$ is assumed to be a unique solution of the SDE (1) which consists of a slowly varying

component governed by the drift coefficient $\emptyset(\cdot)$ and a rapidly fluctuating random component characterized by the diffusion coefficient $\Phi(\cdot)$.

The SDE can be written in integral form as

$$X_t = X_0 + \int_{t_0}^{t} \phi(X_s)ds + \int_{t_0}^{t} \Phi(X_s)dB_s \quad (2)$$

The first integral of the SDE (2) represent the Lebesque integral and the second integral is the Itô or Stratonovich stochastic integral with respect to the Brownian motion or Wienner process $B = \{B_t, 0 \leq t \leq T\}$which is a real-valued continuous process. A solution $\tilde{X}_t$ is said to be unique if any other solution $(\tilde{X}_t)$ is indistinguishable from (X_t), that is

$$\{P(X_t) = (\tilde{X}_t), \forall\, t_0 \leq t \leq T\} = 1$$

For the purpose of this work we assume that equation (2) refers to the Itô form. The Stratonovich SDE can be denoted by the symbol $\circ$ in front of the dB_s By means of a simple tranformation, one can move between the two caculi. The solution of the Itô equation (1) can be written as the solution of the Stratonovich equation

$$dX_t = \bar{\phi}(X_t)dt + \phi(X_t) \circ dB_t \quad (3)$$

which has the modified drift function

$$\bar{\emptyset}(X) = \emptyset(X) - \frac{1}{2}\Phi(X)\Phi'(X), \quad (4)$$

assuming Φ' exists. The Stratonovich calculus follows the usual rules of deterministic calculus, whereas the Itô calculus conveniently relates to martingale theory but has its own stochastic chain rule, the Itô formula. This states that for a twice differentiable function f of the solution of (1) we have

$$df(X_t) = \left(f'(X_t)\phi(X_t) + \frac{1}{2}f''(X_t)\phi^2(X_t)dt\right) + f'(X_t)\phi(X_t)dB_t \quad (5)$$

for $0 \leq t \leq T$. For the case of additive noise the two forms of stochastic calculus coincide (note that a constant drift term will cause the second term to vanish in the transformation (4)). For the most part of this work, we concentrate on the Itô form, as much of the literature is based on this form of calculus. All of the above statements can be extended to higher dimensions, but for simplicity the basic theory is outlined in one dimension.

Kloeden and Pearson (1977), proposed a method for the numerical solution of Itô stochastic differential equations by means of a second-order Runge-Kutta iterative scheme rather than the less efficient Euler iterative scheme. It requires the Runge-Kutta iterative scheme to be applied to a different stochastic differential equation obtained by subtraction of a correction term from the given one. It was observed that different iterative schemes for the numerical solution of stochastic differential equations converge to different solutions for the same noise sample and initial condition. This is in contrast to their deterministic counterparts for ordinary differential equations, which converge to the same solution.

Kloeden and Platen (1992), have discussed extensively about numerical solution of stochastic differential equations in detail. Platen (1999), buttressed this with the discrete time strong and weak approximation methods for the numerical methods to get the solution of stochastic differential equation

The aim of the work is to review and understand better the various techniques

necessary for numerical solutions involving stochastic differential equations and compare the numerical solutions of the Itô form by means of second order Runge-Kutta iterative scheme which is more efficient with a high rate of convergence than the less efficient Euler-Maruyama iterative scheme and the Milstein scheme.

2 Stochastic Taylor Expansion

Consider a 1-dimensional ODE.

$$\frac{d}{dt}X_t = \emptyset(X_t) \tag{6}$$

with initial value X_{t_0}, for $t \epsilon[t_0, T]$ where $0 \leq t_0 \leq T$. The equivalent integral form of equation (6) can be written as:

$$X_t = X_{t_0} + \int_{t_0}^{t} \phi\,(X_s)ds \tag{7}$$

provided $\phi(X_s)$ is smooth enough and have a linear growth bound.

Let $f: \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function, then by chain rule we have:

$$\frac{d}{dt}f(X_t) = \phi(X_t)\frac{\partial}{\partial x}f(X_t), \tag{8}$$

using the operator, say,

$$\mathcal{D} = \phi\frac{\partial}{\partial x}, \tag{9}$$

integrating equation (8), we have

$$f(X_t) = f\,\phi(X_t) + \int_{t_0}^{t} \mathcal{D}f(X_s)ds, \forall\, t\epsilon\left[t_0, T\right] \tag{10}$$

In particular, when $f(x) \equiv x$, we have

$\mathcal{D}f = \phi,$
$\mathcal{D}^2 f = \mathcal{D}(\mathcal{D}f) = \mathcal{D}\phi,$
$\mathcal{D}^3 f = \mathcal{D}(\mathcal{D}^2 f) = \mathcal{D}(\mathcal{D}\phi) = \mathcal{D}^2\phi,$
$\mathcal{D}^4 f = \mathcal{D}(\mathcal{D}^3 f) = \mathcal{D}(\mathcal{D}^3\phi) = \mathcal{D}^3\phi,$

.
.
.

$\mathcal{D}^n f = \mathcal{D}^{n-1}\phi$

and therefore equation (10) reduces to

$$X_t = X_{t_0} + \int_{t_0}^{t} \phi\,(X_s)ds \tag{11}$$

that is to equation (7).

Considering the relation (10) to the function $f = \phi$ in the integral of (11), we have

$$X_t = X_{t_0} + \int_{t_0}^{t} \left(\phi(X_s) + \int_{t_0}^{s} \mathcal{D}\phi\,(X_z) \right) ds$$

$$X_t = X_{t_0} + \phi(X_s) \int_{t_0}^{s} ds + \int_{t_0}^{t} \int_{t_0}^{s} \mathcal{D}\phi\,(X_z) dz ds \tag{12}$$

Again, applying (10) to the function $f = \mathcal{D}\phi$ in the double integral, we have

$$\mathcal{D}\phi(X_z) = \mathcal{D}\phi(X(t_0)) + \int_{t_0}^{z} \mathcal{D}^2\phi(X_u) du$$

so that

$$X_t = X_{t_0} + \phi(X_s) \int_{t_0}^{s} ds + \mathcal{D}\phi(X_{t_0}) \int_{t_0}^{t} \int_{t_0}^{s} dz ds + R_3 \tag{13}$$

with remainder

$$R_3 = \int_{t_0}^{t} \int_{t_0}^{s} \int_{t_0}^{z} \mathcal{D}^2\phi(X_u) du dz ds \tag{14}$$

for $t \in [t_0, T]$. For a general $r + 1$ times continuously differentiable function $f: \mathbb{R} \to \mathbb{R}$ this method gives the classical Taylor formula in integral form:

$$f(X_t) = f(x_{t_0}) + \sum_{k=1}^{r} \frac{(t-t_0)^k}{k!} \mathcal{D}^k f(X_{t_0}) + \int_{t_0}^{t} \dots \int_{t_0}^{s_0} \mathcal{D}^{r+1} f(X_{s_1}) ds_1 \dots ds_{r+1} \tag{15}$$

for $t \in [t_0, T]$ and $r = 1,2,3, \dots$ .The Taylor formula (15) has proven to be a very useful tool in both theoretical and practical investigations, particularly in numerical analysis (Kloeden and Platen 1991). In ordinary differential equations, much of the deterministic numerical analysis is based on manipulating and truncating Taylor expansions.

Analogously, for SDEs we use a stochastic Taylor expansion, with different versions corresponding to the Itô and Stratonovich forms of stochastic calculus. We present the Itô version here as the Itô integrals in the expansion are easier to express in terms of random variables, (Platen, 1999) which form the basis for the numerical examples presented later.

2.1 Itô-Taylor Expansion

The Itô-Taylor expansion is based on repeated iterations of the Itô formula. Consider again the integral equation (2). Note we require the terms $\emptyset$ and Φ to satisfy a linear growth bound and to be sufficiently smooth. For any twice continuously differentiable function $f: R \to R$, Itô's formula gives:

$$f(X_t) = f(X_{t_0}) + \int_{t_0}^{s} \left(\emptyset(X_s) f'(X_s) + \frac{1}{2} \Phi^2(X_s) f''(X_s) \right) ds + \int_{t_0}^{t} \Phi\,(X_s) f'(X_s) dB_s \tag{16}$$

We introduce the operators $\mathcal{D}^0$ and $\mathcal{D}^1$ as

$$\mathcal{D}^0 f = \frac{\emptyset \partial f}{\partial x} + \frac{1}{2} \Phi^2 \frac{\partial^2 f}{\partial x^2} = \emptyset f' + \frac{1}{2} \Phi^2 f''$$

$$\mathcal{D}^1 f = \Phi \frac{\partial f}{\partial x} = \Phi f'$$

Rewriting equation (16) and substituting $\mathcal{D}^0$and $\mathcal{D}^1$, we have

$$f(X_t) = f(X_{t_0}) + \int_{t_0}^{t} \mathcal{D}^0 f(X_s) ds + \int_{t_0}^{t} \mathcal{D}^1 f(X_s)\, dB_s \tag{17}$$

Obviously, for $f(x) \equiv x$, we have $\mathcal{D}^0 f = \emptyset$ and $\mathcal{D}^1 f = \Phi$. In which case, equation (17) reduces to the original Itô equation (2) for X_t. We can apply the Itô formula to the function $f = \emptyset$ and $f = \Phi$ that appear in equation (17) to obtain the following;

$$\emptyset(X_t) = \emptyset(X_{t_0}) + \int_{t_0}^{s} \mathcal{D}^0 \emptyset(X_z) dz + \int_{t_0}^{s} \mathcal{D}^1 \emptyset(X_z)\, dB_z \tag{18}$$

$$\Phi(X_t) = \Phi(X_{t_0}) + \int_{t_0}^{s} \mathcal{D}^0 \Phi(X_z) dz + \int_{t_0}^{s} \mathcal{D}^1 \Phi(X_z)\, dB_z \tag{19}$$

substituting the relations (18) and (19) into the Itô equation (2), we obtain

$$X_t = X_{t_0} + \int_{t_0}^{t} \left[\emptyset(X_{t_0}) + \int_{t_0}^{s} \mathcal{D}^0 \emptyset(X_z) dz + \int_{t_0}^{s} \mathcal{D}^1 \emptyset(X_z)\, dB_z \right] ds + \int_{t_0}^{t} \left[\Phi(X_{t_0}) + \int_{t_0}^{s} \mathcal{D}^0 \Phi(X_z) dz + \int_{t_0}^{s} \mathcal{D}^1 \Phi(X_z)\, dB_z \right] dB_s$$

$$= X_{t_0} + \emptyset(X_{t_0}) \int_{t_0}^{t} ds + \int_{t_0}^{t} \int_{t_0}^{s} \mathcal{D}^0 \emptyset(X_z) dz ds + \int_{t_0}^{t} \int_{t_0}^{s} \mathcal{D}^1 \emptyset(X_z)\, dB_z ds + X_{t_0} + \Phi(X_{t_0}) \int_{t_0}^{t} dB_s + \int_{t_0}^{t} \int_{t_0}^{s} \mathcal{D}^0 \Phi(X_z) dz dB_s + \int_{t_0}^{t} \int_{t_0}^{s} \mathcal{D}^1 \Phi(X_z)\, dB_z dB_s$$

$$= X_{t_0} + \emptyset(X_{t_0}) \int_{t_0}^{t} ds + \Phi(X_{to}) \int_{t_0}^{t} dB_s + \mathbf{R} \tag{20}$$

where **R** is the remainder term, which is given as;

$$\mathbf{R} = \int_{t_0}^{t} \int_{t_0}^{s} \mathcal{D}^0 \emptyset(X_z) dz ds + \int_{t_0}^{t} \int_{t_0}^{s} \mathcal{D}^1 \emptyset(X_z)\, dB_z ds + \int_{t_0}^{t} \int_{t_0}^{s} \mathcal{D}^0 \Phi(X_z) dz dB_s + \int_{t_0}^{t} \int_{t_0}^{s} \mathcal{D}^1 \Phi(X_z)\, dB_z dB_s \tag{21}$$

let $f = \mathcal{D}^1 \Phi$, putting into equation (17), we have

$$\mathcal{D}^1 \Phi(X_t) = \mathcal{D}^1 \Phi(X_{t_0}) + \int_{t_0}^{t} \mathcal{D}^0 \mathcal{D}^1 \Phi(X_s) ds + \int_{t_0}^{t} \mathcal{D}^1 \mathcal{D}^1 \Phi(X_s)\, dB_s \tag{22}$$

Substuting equation (22) into equation (20) we have:

$$X_t = X_{t_0} + \emptyset(X_{t_0})\int_{t_0}^{t} ds + \int_{t_0}^{t}\int_{t_0}^{s} \mathcal{D}^0\emptyset(X_z)dzds + \int_{t_0}^{t}\int_{t_0}^{s} \mathcal{D}^1\emptyset(X_z)\,dB_z ds + \Phi(X_{t_0})\int_{t_0}^{t} dB_s$$
$$+ \int_{t_0}^{t}\int_{t_0}^{s} \mathcal{D}^0\Phi(X_z)dzdB_s + \mathcal{D}^1\Phi(X_{t0})\int_{t_0}^{t}\int_{t_0}^{s} dB_z dB_s$$
$$+ \int_{t0}^{t}\int_{t_0}^{s}\int_{t_0}^{u} \mathcal{D}^1\Phi(X_{t_0})dB_z dB_s du + \int_{t0}^{t}\int_{t_0}^{s}\int_{t_0}^{u} \mathcal{D}^0\mathcal{D}^1\Phi(X_{t_0})dB_z dB_s dB_u$$

$$= X_{t_0} + \emptyset(X_{t_0})\int_{t_0}^{t} ds + \Phi(X_{to})\int_{t_0}^{t} dB_s + \mathcal{D}^1\Phi(X_{t0})\int_{t_0}^{t}\int_{t_0}^{s} dB_z dB_s + \mathbf{R_1} \qquad (23)$$

Where $\mathbf{R}_1$ is the remainder term, which is given as;

$$\mathbf{R}_1 = \int_{t_0}^{t}\int_{t_0}^{s} \mathcal{D}^0\emptyset(X_z)dzds + \int_{t_0}^{t}\int_{t_0}^{s} \mathcal{D}^1\emptyset(X_z)\,dB_z ds + \int_{t_0}^{t}\int_{t_0}^{s} \mathcal{D}^0\Phi(X_z)dzdB_s +$$
$$\int_{t_0}^{t}\int_{t_0}^{s} \mathcal{D}^1\Phi(X_z)dB_z dB_s + \int_{t0}^{t}\int_{t_0}^{s}\int_{t_0}^{u} \mathcal{D}^1\Phi(X_{t_0})dB_z dB_s du +$$
$$\int_{t0}^{t}\int_{t_0}^{s}\int_{t_0}^{u} \mathcal{D}^0\mathcal{D}^1\Phi(X_{t_0})dB_z dB_s dB_u \qquad (24)$$

The properties of the Itô -Taylor expansion can further be expressed with the multiple Itô integrals

$$\int_{t_0}^{t} ds, \int_{t_0}^{t} dB_s, \int_{t_0}^{t}\int_{t_0}^{s} dB_z dB_s$$

and a remainder term involving the next multiple Itô integrals, but not with nonconstant integrands. The Itô -Taylor expansion can be interpreted as a generalization of both the Itô formula and the deterministic Taylor formula.

3 STRONG TAYLOR SCHEMES/APPROXIMATIONS

To apply a numerical scheme to the SDE (1), we must first discretize our time interval $[0,T]$, using a fixed step-size $\Delta=T/N$. This gives us a set of partition

$$0 = \tau_0 < \tau_1 < \cdots < \tau_n < \cdots < \tau_N = T,$$

which we use to approximate our solution. We now present some of the theoretical bases of different methods of numerical solution.

3.1 The Euler-Maruyama Method

We examine the stochastic Euler scheme for time discret approximation of Itô processes and use the simulation of approximating time discrete trajectories to handle some typical problems. One of the simplest time discrete approximations of the Itô process is the Euler (or the Euler-Maruyama) Approximation. (Burrage, Burrage, and Mitsui, 2000)
Consider a stochastic process $X = \{X_t\ t_0 \le t \le T\}$ satisfying (1) for $X_t = X_0$. For a given discretization

$$t_0 = \tau_0 < \tau_1 < \cdots < \tau_n < \cdots < \tau_N = T,$$

of the time interval $[t_0\ T]$, an Euler approximation is a continuous time stochastic process $Y = \{Y(t),\ t_0 \le t \le T\}$ satisfying the iterative scheme

$$Y_{n+1} = Y_n + \emptyset(Y_n)\Delta_n + \Phi(Y_n)\Delta B_n \qquad (25)$$

with initial value $Y_0 = X_0$ where $Y_n = Y(\tau_n), \Delta_n = \tau_{n+1} - \tau_n$ and $\Delta B_n = B\tau_{n+1} - B\tau_n$ for all $n = 0,1,2,\cdots,N-1$, of the Wiener process $B = \{B_t\ t \geq 0\}$. From definition of the Wiener process, it follows that these increments are independent Gaussian random variables with mean $\mathbb{E}(\Delta B_n) = 0$ and variance $\mathbb{E}((\Delta B_n)^2) = \Delta_n$

In general, the sample paths of an Itô process inherit the irregularity of the sample paths of its driving Wiener process and in particular, their non-differentiability. In examining the first three terms of the stochastic Taylor expansion, we see that these form the basis of the Euler-Maruyama scheme upon evaluating the integrals, yielding Δ and ΔB respectively, (Kloeden and Platen, 1992). For the case where the diffusion term $\Phi \equiv 0$ this reduces to the ordinary deterministic Euler scheme. The Euler-Maruyama method converges with strong order $\alpha = \frac{1}{2}$. For drift terms the Euler-Maruyama scheme has strong order $\alpha = 1$ For most other cases, however the method provides a poor estimate of the solution, particularly in cases where the coefficients are non-linear, as is well documented in the deterministic Euler case. For more satisfactory levels of accuracy higher order schemes are required. (Mahony, 2006).

3.2 Milstein Method

Consider the Itô -Taylor expansion (23) and add to the Euler-Maruyama scheme (25) the additional term

$$\Phi(X_{to})\Phi'(X_{to})\int_{t_0}^{t}\int_{t_0}^{s} dB_z dB_s = b(X_{t_0})\Phi'(X_{t_0})\mathbb{I}_{(1,1)}$$
$$= \Phi(X_{to})\Phi'(X_{to})\frac{1}{2}((\Delta B)^2 - \Delta))$$

then we obtain the Milstein scheme

$$Y_{n+1} = Y_n + \Phi(Y_n)\Delta + \Phi(Y_n)\Delta B + \frac{1}{2}\Phi(Y_n)\Phi'(Y_n)((\Delta B)^2 - \Delta)) \quad (26)$$

which has strong order of convergence α =1.0. Thus, with the addition of just one more term to the Euler scheme to form the Milstein scheme we increase the strong convergence order from α=0.5 *to* α=1.0. For Φ=0, the strong order α=1.0 of the Milstein scheme corresponds to that of the Euler scheme in the deterministic case. The additional term marks the point of divergence of stochastic numerical analysis from deterministic, (Milstein 2016). Therefore, the Milstein scheme is regarded as the proper generalization of the deterministic Euler scheme for strong convergence criterion because it gives the same order of strong convergence as the deterministic case.

3.3 Strong Taylor Scheme

The Euler-Maruyama scheme and the Milstein scheme can be considered to be specific cases of the more general class of methods known as strong Taylor approximations formed by including appropriately many terms from stochastic-Taylor expansions. In principle, arbitrarily many terms can be added to create schemes of a desired level of convergence. Khodabin, et. al., 2011). For practical implementation, however, this is at the expense of evaluating more and more derivatives and stochastic integrals, leading to expressions that become very complicated as the desired order of

convergence increases, Kloeden, and Platen (1991), Consider, for example, the Taylor order 1.5 scheme for the SDE (1.1).

$$Y_{n+1} = Y_n + \emptyset(Y_n)\Delta + \Phi(Y_n)\Delta B + \Phi(Y_n)\Phi'(Y_n)\mathbb{I}_{1.1} + \Phi(Y_n)\emptyset'(Y_n)\mathbb{I}_{(1,0)} + \frac{1}{2}\Delta^2\left(\emptyset(Y_n)\emptyset'(Y_n) + \frac{1}{2}\Phi^2(Y_n)\emptyset''(Y_n)\right) + \left(\emptyset(Y_n)\Phi'(Y_n) + \frac{1}{2}\Phi^2(Y_n)\Phi''(Y_n)\right)\mathbb{I}_{(0,1)} + \Phi(\Phi(Y_n)\Phi''(Y_n) + (\Phi'(Y_n))^2)\mathbb{I}_{(1,1,1)}$$

which, upon evaluation of the integrals becomes

$$Y_{n+1} = Y_n + \emptyset(Y_n)\Delta + \Phi(Y_n)\Delta B + \frac{1}{2}\Phi(Y_n)\Phi'(Y_n)((\Delta B)^2 - \Delta) + \Phi(Y_n)\Phi'(Y_n)\Delta Z + \frac{1}{2}\Delta^2\left(\emptyset(Y_n)\emptyset'(Y_n) + \frac{1}{2}\Phi^2(Y_n)\emptyset''(Y_n)\right) + \left(\emptyset(Y_n)\Phi'(Y_n) + \frac{1}{2}\Phi^2(Y_n)\Phi''(Y_n)\right)(\Delta B\Delta - \Delta Z) + \Phi(\Phi(Y_n)\Phi''(Y_n) + (\Phi'(Y_n))^2)\left(\frac{1}{3}(\Delta B)^2 - \Delta\right)\Delta B \quad (27)$$

The approach above, becomes computationally more difficult as higher order methods are required. Clearly, one of the principal problems in the practical implementation of higher order Taylor approximations is that derivatives of higher orders have to be evaluated. To this end, we consider another class of strong approximation methods known as stochastic Runge-Kutta schemes.

3.4 Runge-Kutta Schemes

The deterministic Runge-Kutta schemes cannot be easily adapted to an SDE, because they only converge with a given strong order towards the correct solution if they also approximate the corresponding strong Taylor scheme, Milstein and Tretyakov (2005). Consider the one dimensional case $d = m = 1$ the implicit order 1.0 strong Runge-Kutta scheme is

$$Y_{n+1} = Y_n + \emptyset(\tau_{n+1}, Y_{n+1})\Delta + \Phi\Delta B + \frac{1}{2\sqrt{\Delta}}\left(\Phi(\tau_n, \tilde{\gamma}) - \Phi\right)\{(\Delta B)^2 - \Delta\} \quad (28)$$

with $\tilde{\gamma}_n = Y_n + \Phi(Y_n)\sqrt{\Delta}$

This method is essentially an approximation of the Milstein scheme, and was one of the first of a body of methods that avoid the need to calculate derivatives, which is of particular importance when implementing a method on computer.

Order 1.0 Strong Runge-Kutta Scheme

This scheme converges with strong order $\alpha = 1.0$ and are derivatives free and generally appeared simply in their Stratonovich form.

Consider the following method:

$$y_1 = Y_n$$
$$y_2 = Y_n + \emptyset(y_1)\Delta + \Phi(y_1)\Delta B,$$
$$Y_{n+1} = Y_n + \Delta\emptyset(y_1) + \Delta B\left(\frac{1}{2}\Phi(y_1) + \frac{1}{2}\Phi(y_2)\right) \quad (29)$$

Similarly, we have the method known as two-step Runge-Kutta due to Burrage (1993) which has an identical structure:

$$y_1 = Y_n$$
$$y_2 = Y_n + \frac{2}{3}\emptyset(y_1)\Delta + \frac{2}{3}\Phi(y_1)\Delta B,$$

$$Y_{n+1} = \mathrm{Y_n} + \Delta\left(\frac{1}{4}\emptyset(\mathcal{Y}_1) + \frac{3}{4}\emptyset(\mathcal{Y}_2)\right) + \Delta\mathrm{B}\left(\frac{1}{4}\Phi(\mathcal{Y}_1) + \frac{3}{4}\Phi(\mathcal{Y}_2)\right) \quad (30)$$

And finally the method due to Higham, (2013) given by

$$\begin{aligned}\mathcal{Y}_1 &= Y_n \\ \mathcal{Y}_2 &= Y_n + \frac{1}{2}\Phi(\mathcal{Y}_1)\Delta B, \\ Y_{n+1} &= \mathrm{Y_n} + \Delta\emptyset(\mathcal{Y}_1) + \Delta\mathrm{B}\big(\Phi(\mathcal{Y}_1)\big)\end{aligned} \quad (31)$$

All of these algorithms will certainly provide a good first approximation to the solution of an SDE, and are relatively easy to implement compared to higher-order Taylor schemes such as (27).

4 Strong And Weak Convergence

The solutions of stochastic differential equation are not explicitly known, so we try to discover them by simulation. If the solutions are known explicitly, then we can calculate the error of an approximation using the absolute error criterion, i.e. the expectation of the absolute value of the difference between the approximation and the Itô process at the time T, that is

$$\varepsilon = \mathbb{E}(|X_T - Y(T)|), \quad (32)$$

which gives the measure of the pathwise closeness at the end of the time interval [0, T]. When the diffusion coefficient $\Phi \equiv 0$ and the initail value is deterministic, randomness has no effect and the expectation in (32) is suficient. The criterion (32) then reduces to the deterministic absolute error criterion (global truncation error). (Burrage, 1995).

4.1 Strong Convergence

The concept of strong convergence uses the concept of the absolute error criterion (32). We say that a discrete time approximation Y^δ with maximum step size δ converges strongly to the exact solution X with order $\alpha > 0$ at time T if

$$\lim_{\delta \downarrow 0} \mathbb{E}\big(\big|X_T - Y^\delta(T)\big|\big) = 0 \quad (33)$$

In order to assess and compare different time discrete approximations, we need to know their rates of strong convergence. We say that a discrete time approximation Y^δ converges strongly with order $\alpha > 0$ at time T if there exists constant $K < \infty$ which does not depend on δ, and $\delta_0 > 0$ such that

$$\varepsilon(\delta) = \mathbb{E}\big(\big|X_T - Y^\delta(T)\big|\big) \leq \mathcal{K}\delta^\alpha \quad (34)$$

for each $\delta \in (0, \delta)$ and for sufficiently stepsize δ. This definition generalizes the standard convergence caterion for ordinary differential equations. Although the Euler method for ordinary differential equations has order 1, while the strong order of Euler-Murayama method for SDE is of order $\frac{1}{2}$. This fact was proved in Higham (2000).

4.2 Weak Convergence

We say that a general time discrete approximation Y^{δ} corresponding to a time discretization $(\tau)_{\delta}$ converges weakly to X at time T as $\delta \downarrow 0$ with respect to a class C of the text function $g: \mathbb{R}^d \rightarrow \mathbb{R}$ if we have

$$\lim_{\delta \downarrow 0} \mathbb{E}\left|(g(X_T)) - \mathbb{E}(g\left(Y^{\delta}(T)\right))\right| = 0 \qquad (35)$$

for all $g \in C$. If K contains all polynomials, this definition implies the convergence of all moments, so theoretical investigations involving it will require the existence of all moments. In the deterministic case with a zero diffusion coefficient and a non-random initial value, equation (35) with $g(x) \equiv x$ reduces to the usual deterministic convergence criterion, just as the strong convergence criterion (33) does.

We say that a time discrete approximation Y converges weakly with order $\gamma > 0$ to X at time T as $\delta \downarrow 0$ if for each polynomial g, there exists a positive constant C, which does not depend on δ and a finite $\delta_0 > 0$ such that

$$\mathbb{E}\left|(g(X_T)) - \mathbb{E}(g\left(Y^{\delta}(T)\right))\right| \leq C\delta^{\gamma} \qquad (36)$$

for each $\delta \in (0, \delta_0)$. The Euler approximation usually converges with weak order β=1, in contrast with the strong order γ =0.5.

The strong and weak convergence criteria lead to the development of different time discrete approximations which are only efficient with respect to one of the two criteria. This fact makes it important to clarify the aim of a simulation before choosing an approximation scheme.

5 Weak Approximation Methods

As we already mentioned, it is not always necessary to simulate individual trajectories of a solution of a SDE, sometimes only information about moments or other functionals may be all that is required, in which case weak convergence is all that is required. Within this section we discuss numerical methods that focus on approximating the probability distributions of solutions of SDEs, allowing us to handle wide classes of functionals. We then need to study the weak order of convergence of several stochastic numerical methods.

5.1 Weak Euler Approximation

The weak convergence criterion (36) allows us more degrees of freedom in constructing a discrete time approximation than the strong convergence criterion (34). For instance, under weak convergence, the random increments, ΔB of the Brownian motion can be replaced by simpler random variables, $\tilde{\Delta} B$ which are similar to these in distribution. By substituting the $N(0,\Delta)$ Gaussian distributed random variable ΔB in the Euler approximation (25) by an independent two-point distributed random variable $\tilde{\Delta} B$ with

$$\mathbb{P}\left(\Delta \tilde{B} = \pm\sqrt{\Delta}\right) = \frac{1}{2} \qquad (37)$$

we obtain the simplified Euler method

$$\mathcal{Y}_{n+1} = Y_n + \emptyset(Y_n)\Delta_n + \Phi(Y_n)\Delta\tilde{B}_n \qquad (38)$$

The main focus for this choice of the two-point random variable $\tilde{\Delta} B$ is that its first two moments match the corresponding ones for ΔB. It can be shown that, under a sufficiently

regularity conditions, this method (38) converges with weak order β=1.0. This weak order is higher than the strong order γ=1.0 achieved by the Euler approximation (25).

5.2 Weak Taylor Scheme

As with strongly convergent scheme, we can derive more accurate weak Taylor schemes by including the multiple stochastic integrals from the stochastic Taylor expansion. However, the objectives is to obtain more information about the probability measure of the underlying Itô process, rather than about its sample paths.

If we want to construct a weak Taylor scheme of order 2.0 in the Itô case all terms with single and double integrals need to be included from the Itô-Taylor expansion, resulting in the following scheme

$$Y_{n+1} = Y_n + \emptyset(Y_n)\Delta + \Phi(Y_n)\Delta B + \Phi(Y_n)\Phi'(Y_n)\mathbb{I}_{(1,1)} + \Phi(Y_n)\emptyset'(Y_n)\mathbb{I}_{(1,0)} + \frac{1}{2}\Delta^2\left(\emptyset(Y_n)\emptyset'(Y_n) + \frac{1}{2}\Phi^2(Y_n)\emptyset'(Y_n)\right) + \left(\emptyset(Y_n)\Phi'(Y_n) + \frac{1}{2}\Phi^2(Y_n)\Phi''(Y_n)\right)\mathbb{I}_{(0,1)}$$

which, upon evaluation of the integrals becomes

$$\begin{aligned} Y_{n+1} = Y_n + \emptyset(Y_n)\Delta + \Phi(Y_n)\Delta B + \frac{1}{2}\Phi(Y_n)\Phi'(Y_n)((\Delta B)^2 - \Delta) + \Phi(Y_n)\Phi'(Y_n)\Delta Z \\ + \frac{1}{2}\Delta^2\left(\emptyset(Y_n)\emptyset'(Y_n) + \frac{1}{2}\Phi^2(Y_n)\emptyset''(Y_n)\right) \\ + \left(\emptyset(Y_n)\Phi'(Y_n) + \frac{1}{2}\Phi^2(Y_n)\Phi''(Y_n)\right)(\Delta B\Delta - \Delta Z) \end{aligned} \quad (39)$$

We still obtain a scheme of weak order β = 2.0 if we replace the random variable ΔB in (39) by $\Delta\tilde{B}$. Here $\Delta\tilde{B}$ might be a three-point distributed random variable with

$$\mathbb{P}\left(\Delta\tilde{B} = \pm\sqrt{3\Delta}\right) = \frac{1}{6} \quad \text{and } \mathbb{P}\left(\Delta\tilde{B} = 0\right) = \frac{2}{3} \quad (40)$$

Note that the first four moments of $\Delta\tilde{B}$ match the corresponding ones of ΔB. By approximating all triple Itô integrals in the order β=3.0 weak Taylor scheme, a simplified order 3.0 weak Taylor scheme was derived by Platen (1984), with the form

$$\begin{aligned} Y_{n+1} = Y_n + \emptyset(Y_n)\Delta + \Phi(Y_n)\Delta B + \frac{1}{2}\Phi\Phi'\left((\Delta\tilde{B})^2 - \Delta\right) + \Phi\emptyset'\Delta\tilde{Z} + \frac{\Delta^2}{2}\left(\emptyset\emptyset' + \frac{1}{2}\Phi^2\Phi''\right) \\ + \left(\emptyset\Phi' + \frac{1}{2}\Phi^2\Phi''\right)(\Delta\tilde{B}\Delta - \Delta\tilde{Z}) \\ + \left(\emptyset\left(\emptyset\Phi' + \Phi\emptyset' + \frac{1}{2}\Phi^2\Phi''\right) + \frac{1}{2}\Phi^2\left(\emptyset\Phi' + \Phi\emptyset' + \frac{1}{2}\Phi^2\Phi''\right)\right. \\ \left. + \Phi\left(\emptyset\emptyset' + \frac{1}{2}\Phi^2\Phi''\right)'\right)\left(\frac{\Delta\tilde{B}\Delta^2}{6}\right) \\ + \left(\Phi(\Phi\emptyset' + \emptyset\Phi' + \frac{1}{2}\Phi^2\Phi'')' + \emptyset(\Phi\Phi') + \frac{1}{2}\Phi^2(\Phi\Phi')''\right)\left(\frac{(\Delta\tilde{B})^2 - \Delta}{6}\right) \\ + \left(\emptyset(\emptyset\emptyset' + \frac{1}{2}\Phi^2\Phi'')' + \frac{1}{2}\Phi^2(\emptyset\emptyset' + \frac{1}{2}\Phi^2\Phi'')''\right)\frac{\Delta^3}{6} \\ + \Phi(\Phi\Phi')\left((\Delta\tilde{B})^2 - 3\Delta\right)\frac{\Delta\tilde{B}}{6} \end{aligned} \quad (41)$$

Here ΔB and $\Delta\tilde{B}$ can be chosen, for instance, as correlated zerp mean Guassian random varibles with

$$\mathbb{E}\left(\Delta\tilde{B}\right)^2 = \Delta, \qquad \mathbb{E}\left(\left(\Delta\tilde{Z}\right)^2\right) = \frac{\Delta^3}{3}, \qquad \mathbb{E}(\Delta\tilde{Z}\Delta\tilde{B}) = \frac{\Delta^2}{2} \tag{42}$$

We note that the weak higher-order Taylor schemes involve higher-order derivatives and it would be desirable to have derivative free or Runge-Kutta-type weak schemes.

5.3 Weak Runge-Kutta Methods

A weak second-order Runge-Kutta approximation that avoids derivatives in drift and the diffusion coefficients is given by the algorithm

$$Y_{n+1} = \mathrm{Y_n} + \left(\emptyset\left(\bar{Y}_n\right) + \emptyset(Y_n)\right) + \frac{\Delta}{2} + \left(\Phi\left(\mathrm{Y_n}^+\right) + \Phi(\mathrm{Y_n}^-)\right)\frac{\Delta\widetilde{B_n}}{4} +$$
$$\left(\Phi\left(\mathrm{Y_n}^+\right) - \Phi(\mathrm{Y_n}^-)\right)\left(\left(\Delta\widetilde{B_n}\right)^2 - \Delta\right)\frac{1}{4\sqrt{\Delta}} \tag{43}$$

with

$$\bar{\mathrm{Y}}_n = \mathrm{Y_n} + \emptyset\left(\bar{Y}_n\right)\Delta + \Phi(Y_n)\Delta\tilde{B}_n$$

and

$$\mathrm{Y_n}^{\pm} = \mathrm{Y_n} + \emptyset(Y_n)\Delta + \Phi(Y_n)\sqrt{\Delta}$$

- The random variable $\Delta\tilde{B}$ can be chosen as before in (43)

Weak Runge-Kutta type methods are not as well developed as other areas of the numerical analysis of SDEs. It is still a relatively open-ended area, with many developments taking place only recently.

6 Some Numerical Results and Implementation Issues

We now address some of the practical issues arising from the usage of some of the numerical schemes mentioned by presenting their numerical results and their implementations. It is informative to have an equation with a known solution so that the accuracy of a numerical scheme can be gauged. To this end we, introduce a linear test equation. This equation is often used as it has multiplicative noise, has an explicit solution, and is used to model asset prices in financial mathematics and it is of the Itô form

$$dX_t = \lambda X_t dt + \mu X_t dB_t, \qquad X(0) = X_0 \qquad 0 \le t \le T \tag{44}$$

and has the explicit solution

$$X_t = X_0 \exp\left(\left(\lambda - \frac{1}{2}\mu^2\right)t + \mu B_t\right) \tag{45}$$

for $t\in[0,T]$ and the given Brownian motion. $B = (B_t\ t \ge 0)$

We first consider a number of strong solution methods and compare their numerical solution with the explicit one, for an individual sample path of the solution. We use a random number generator that produces normally distributed random numbers, with zero mean and variance equal to the step-size of the discretized time interval when generating the exact solution. When comparing a numerical approximation to an individual path of the explicit solution we must use this same set

of random numbers in order to keep the approximation and the path consistent.

By interpreting the results given by a scheme that uses a random number generator we are assuming that errors in the generator, or random number bias, are negligible. To generate an increment with *mean*=0 and *Variance* =Δ we need simply to generate a normally distributed number and multiply it by $\sqrt{\Delta}$. For the additional random variable ΔZ in (41), we must satisfy properties (42) using the following fact: If G_1 *and* G_2 are two independent Gaussian random variables, then

$$\Delta W = \sqrt{\Delta} \text{ and } \Delta Z = \frac{1}{2}\Delta^{\frac{2}{3}}(G_1 + \frac{1}{\sqrt{3}}G_2) \qquad (46)$$

form a pair of random variables with the appropriate properties. For a strong order 2.0 Taylor scheme as found in Akinbo, Faniran, and Ayoola (2015) these same increments are used.

The strong versions of the Euler-Maruyama scheme, the Milstein scheme, the Taylor order 1.5 scheme and the Runge-Kutta method due to Platen (1993) were implemented to produce the following graphic, which gives a visual guide to the accuracy of the schemes. The methods were implemented over T=1, with time-step $\Delta=2^{-4}$, parameter values λ=2, μ=1 and initial condition X_0=1, to produce Figure 1.

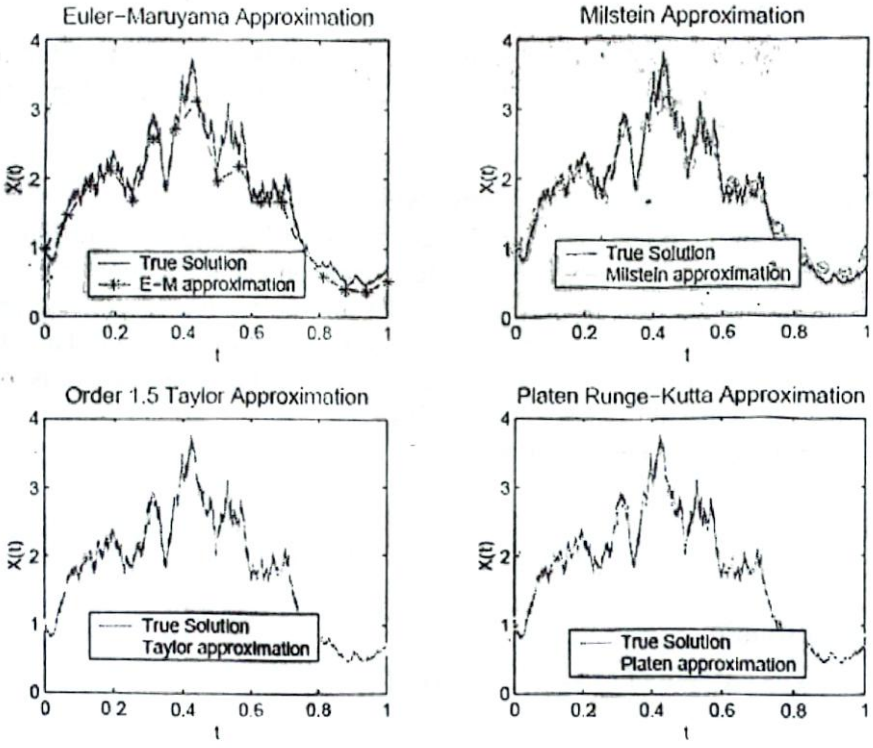

Figure 1: Path-wise accuracy of numerical schemes with step-size $\Delta=2^{-4}$.

The schemes seem to be consistent with the their strong order convergence measures, i.e. the strong order 1.5 Taylor scheme appears to be the best approximation, while the Euler method (strong order 0.5) appears to be the poorest. If we consider the same parameter values but a finer time-step $\Delta=2^{-7}$ we can see the results in Figure 2.

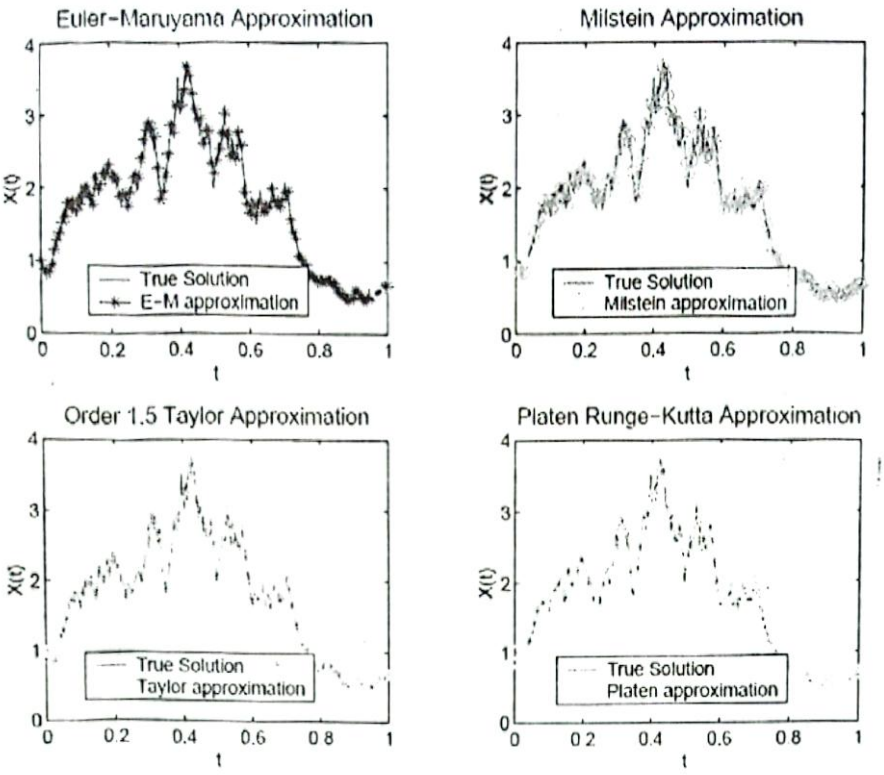

Figure 2: Path-wise accuracy of numerical schemes with step-size $\Delta=2^{-7}$.

As the step-size becomes smaller the numerical solution appears to match the true solution more accurately, as indicated in Figure 2. In order to test this strong convergence we can consider how the error at the endpoint is dependent upon the step-size and plot the results on a log-log scale. The resulting plot should be a straight line of slope equal to that of the order of strong convergence. This idea was verified for the same four methods checked above and ensure that the slopes are correct sample lines of the appropriate slope were plotted for comparison. Figure 3 indicates the results. The correct orders of 0.5, 1.0, 1.5 and 1.0 all appear to be verified. Further confirmation by using a least squares fit for the appropriate power-law was also calculated

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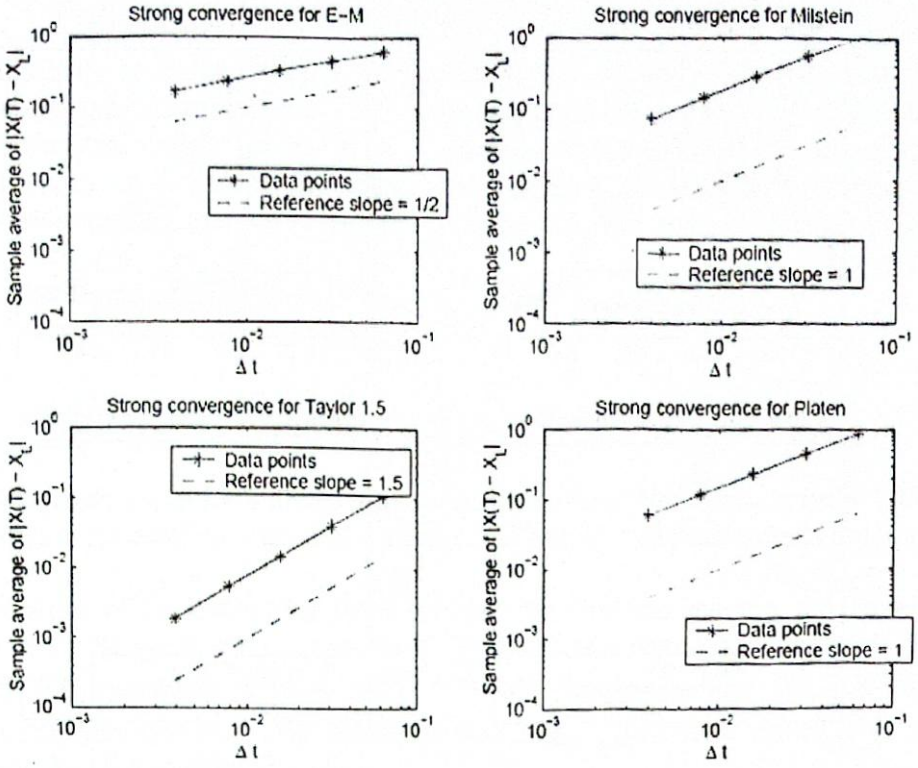

Figure 3: Strong Convergence for Various Numerical Methods.

We present the results of using the above methods to calculate the mean of the solution of (44). While we see the solution is non-smooth along individual trajectories (for λ=2 and μ=1), as indicated by Figures 1 and 2, the mean of the solution is a smooth plot. Figure 4 indicates the results of ensemble averaging over 1000 experiments, where the true average is plotted also. The length of time required for each calculation was dependent upon the complexity of the algorithm, as expected.

To illustrate the advantage of using weak order schemes with simpler random variables, we can compare the results of computing the mean of the solution of (44) with a weak method versus a strong method. As a simple example we consider the Euler scheme. The weak and strong versions of this method have an identical structure but different random variables are used to generate the appropriate Weiner increments. The expected value of the solution was computed in both cases and compared with the true expected value as shown in Figure 5.

The advantages of using a weak scheme for certain purposes are clear; similar results to strong methods are produced but the scheme is computationally much simpler. Much time would we wasted unnecessarily computing moments of a solution using strong order schemes. The weak order 2.0 Taylor scheme (42) can be used to approximate the expected value of the solution in a similar manner. As expected, the scheme provides a much better approximation of the true mean of the solution, with a very small endpoint error that decreases with the step-size as expected

.

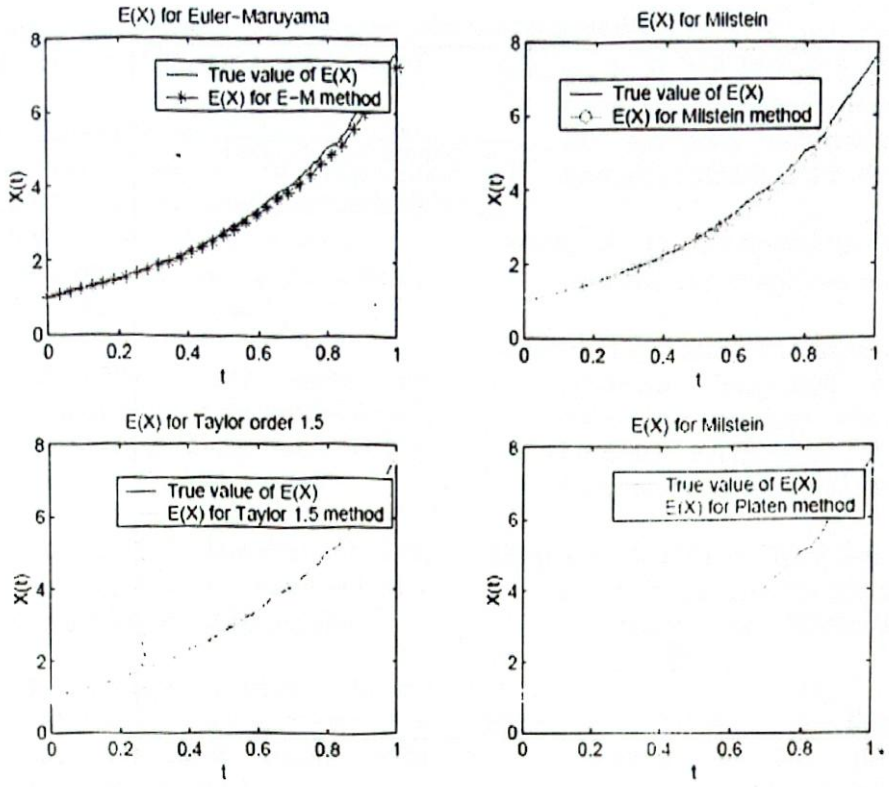

Figure 4: $E(X)$ for Solution of (44)

The question of stability for this equation has been well analysed in Higham (2013). Upon examining the solution, we can see that the questions of mean-square and asymptotic stability can be answered in terms of the parameters λ and μ in the explicit solution (45). In stability analysis we usually consider the more general case where these parameters are allowed to be complex. We can characterize m-mean-square stability for the solution by the following:

$$\lim_{t\to\infty} E(|X(t)^2|)=0 \Leftrightarrow Re(\lambda)+\frac{1}{2}|\mu|^2<0 \qquad (47)$$

Similarly the necessary criteria for asymptotic stability for this particular solution is given by the following:

$$\lim_{t\to\infty} E(|X(t)|)=0 \; with Probability\; 1 \Leftrightarrow Re+\frac{1}{2}|\mu|^2<0 \qquad (48)$$

To examine mean-square stability for the Euler-Maruyama method, we can solve (44) numerically over [0; 20] for mean-square stable parameter values and different time-steps. We perform 500000 experiments for the time-steps $\Delta=1, \frac{1}{2}, and\ \frac{1}{4}$. Only $\Delta=\frac{1}{4}$ appears to provide the desired behavior, as shown in Figure 5. To examine asymptotic stability, we can compute a single Brownian path over a long time period [0,5000] with the same step-sizes and asymptotically stable parameter values. Here we see that again only $\Delta=\frac{1}{4}$ produces a path that decays to zero as desired.

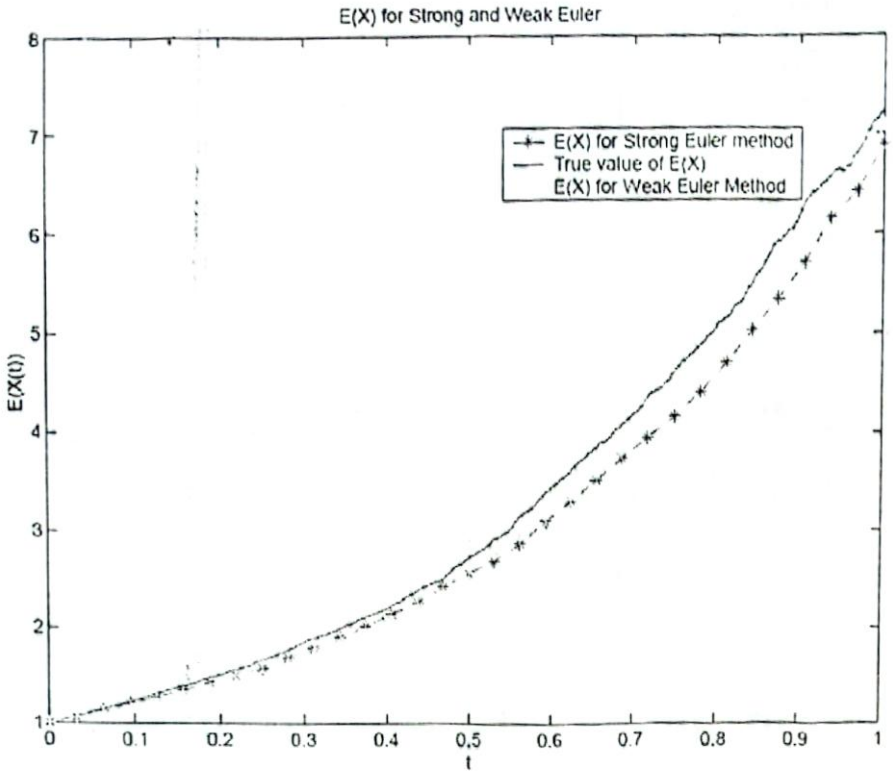

Figure 5: $E(X)$ as computed by strong and weak Euler methods

7 Conclusions

Numerical methods for the solution of stochastic differential equations are essential for the analysis of random phenomena. Strong solvers are necessary when exploring characteristics of systems that depend on trajectory-level properties. Several approaches exist for strong solvers, in particular Taylor and Runge-Kutta type methods, although both increase greatly in complication for orders greater than one.

This paper has discussed some techniques for exploring the behavior of stochastic differential equation, taking into consideration the Brownian Motion which served as a basis in finance for computing the expected path of a function of stochastic process. We review some numerical techniques for solving stochastic differential equation (SDEs) such as the Euler-Maruyama, Milstein, Taylor and Runge-Kutta methods.

We finally performed some convergences analysis and found out that the weak order 2.0 Taylor scheme (42) can be used to approximate the expected value of the solution in a similar manner. The scheme provides a much better approximation of the true mean of the solution, with a very small endpoint error that decreases with the step-size as expected.

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